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Combined Data Envelopment Analysis and Analytical Hierarchy Process Methods to obtain the Favorable Weights from Pairwise Comparison Matrix

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Abstract
Finding the favorable weights from pairwise comparison matrices is very important to rank the alternatives in Multi Attribute Decision Making (MADM) concept. Using Linear Programming has considered to develop this field during recent researches. Combined Analytic Hierarchy Process and Data Envelopment Analysis considered by some researches to find the weights. Decreasing the numerical procedures is vital because of using linear programming in proposed models. In this paper we are going to present some models to generate the favorable weights from a pairwise comparison matrix combing Analytic Hierarchy Process (AHP) and Data Envelopment Analysis (DEA) models. First we address a history and applications of AHP and then some notes on DEA models. Finally three DEA models are proposed to find the weights. In this way we tried to decrease the computational process by decreasing the number of constraints and variables of proposed models. Some numerical examples are putted forward to illustrate our approaches.

Keywords: Data Envelopment Analysis (DEA), Analytic Hierarchy Process(AHP), pairwise comparison matrix.

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1. Introduction

Data envelopment analysis (DEA) is one of the most popular tools in production management literature for performance measurement, while the analytic hierarchy process (AHP) is a popular tool in the field of multiple-criteria decision-making (MCDM). MCDM has been succinctly defined as making decisions in the face of multiple conflicting objectives [9]. Many researchers have found similarities between DEA and MCDM techniques. One of the earliest attempts to integrate DEA with multi-objective linear programming (a MCDM technique) was provided by Golany [5]. Since then, there have been several attempts to use the principles of DEA in the MCDM literature. The traditional goals of DEA and MCDM have been compared by Stewart [8]. The goal of DEA is to determine the productive efficiency of a system or decision-making-unit (DMU) by comparing how well the DMU converts inputs into outputs, while the goal of MCDM is to rank and select from a set of alternatives that have conflicting criteria. It has been recognized for more than a decade that the MCDM and DEA formulations coincide if inputs and outputs can be viewed as criteria for performance evaluation, with minimization of inputs and/or maximization of outputs as associated objectives [2,4].

It is not the intention in this paper to go into the details of similarities. This paper is concerned about using the concepts of DEA for addressing specific problems in the AHP [3], a popular technique of MCDM. The paper begins with brief discussions on DEA and AHP and a literature survey on previous approaches linking the two methods. The proposed approaches synthesizing DEA concepts in AHP is discussed and illustrated in Section 3. A numerical example illustrated in the fourth section to compare the models. The paper ends with a summary and conclusions of the paper.

2. Backgrounds

2.1. Analytic Hierarchy Process

Analytic Hierarchy Process (AHP), since its invention, has been a tool at the hands of decision makers and researchers; and it is one of the most widely used multiple criteria decision-making tools. Many outstanding works have been published based on AHP: they include applications of AHP in different fields such as planning, selecting a best alternative, resource allocations, resolving conflict, optimization, etc., and numerical extensions of AHP. Bibliographic review of the multiple criteria decision-making tools carried out by Steuer [7] is also important. This review paper is partially dedicated to the AHP applications, which are combined with finance. The speciality of AHP is its flexibility to be integrated with different techniques like Linear Programming, Quality Function Deployment, Fuzzy Logic, etc. This enables the user to extract benefits from all the combined methods, and hence, achieve the desired goal in a better way.

Analytic Hierarchy Process is a multiple criteria decision-making tool. This is an Eigne value approach to the pairwise comparisons. It also provides a methodology to calibrate the numeric scale for the measurement of quantitative as well as qualitative performances. The scale ranges from 1 to 9 for least valued than, to 1 for equal, and to 9 for absolutely more important than covering the entire spectrum of the comparison. Some key and basic steps involved in this methodology are:

Step1: Structuring of the decision problem into a hierarchical model.

It includes decomposition of the decision problem into elements according to their common characteristics and the formation of a hierarchical model having different levels. A simple AHP model has three levels (goal, criteria and alternatives), more complex models containing more than three levels are also used in the literature. For example, criteria can be divided further into sub criteria and sub-sub-criteria. Additional levels containing different actors relevant to the problem under consideration may also be included in AHP studies.

Step 2: Making pairwise comparisons and obtaining the judgment matrix.

In this step, the elements of a particular level are compared with respect to a specific element in the immediate upper level. The resulting weights of the elements may be called the local weights. The opinion of a decision maker (DM) is elicited for comparing the elements. Elements are compared pairwise and judgments on comparative attractiveness of elements are captured using a rating scale (1 to 9 scale in traditional AHP). Usually, an element receiving higher rating is viewed as superior (or more attractive) compared to another one that receives a lower rating. The comparisons are used to form a matrix of pair-wise comparisons called the judgment matrix A. Each entry a_{ij} of the judgment matrix are governed by the three rules: $a_{ij} > 0; a_{ij} = 1/a_{ji}$; and $a_{ii} = 1$ for all i. If the transitivity property holds, i.e., $a_{ij} = a_{ik}.a_{kj}$, for all the entries of the matrix, then the matrix is said to be consistent. If the property does not hold for all the entries, the level of inconsistency can be captured by a measure called consistency ratio (see next step).

Step 3: Local weights and consistency of comparisons.

In this step, local weights of the elements are calculated from the judgment matrices using the eigenvector method (EVM). The normalized eigenvector corresponding to the principal eigenvalue of the judgment matrix provides the weights of the corresponding elements. Though EVM is followed widely in traditional AHP computations, other methods are also suggested for calculating weights, including the logarithmic least-square technique (LLST), goal programming, and others. When EVM is used, consistency ratio CR can be computed. For a consistent matrix A value of CR less than 0.1 is considered acceptable because human judgments need not be always consistent, and there may be inconsistencies introduced because of the nature of scale used. If CR for a matrix is more than 0.1, judgments should be elicited once again from the DM till he gives more consistent judgments.

Step 4: Aggregation of weights across various levels to obtain the final weights of alternatives.

Once the local weights of elements of different levels are obtained as outlined in Step 3, they are aggregated to obtain final weights of the decision alternatives (elements at the lowest level).

In this chapter we only focussed on step 2 and finding the weights. The case of consistency may be considered for feature research. We are going to present some DEA models to generate the local weights.

Some notes on DEA models appeared in the next section before presenting the recent work and our proposals to combine DEA and AHP.

2.2. Some Notes on DEA Models

Consider $n, DMUs$ with m inputs and s outputs. The input and output vectors of DMU_j $(j=1,..,n)$ are

$X_j = (x_{1j},\dots,x_{mj})^t, Y_j = (y_{1j},\dots,ysj)^t$

which $X_j \geq 0, X_j \neq 0, Y_j \geq 0, Y_j \neq 0$.

By using the non-empty, constant return to scale, convexity and possibility postulates, the production possibility set (PPS) is made as follows:

$$T_c = \left\{ \begin{array}{l} (X, Y): X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \\ \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j = 1\dots, n \end{array} \right\}$$

Let DMU_o is evaluated. The multiplier form of CCR model [3], in input oriented case is as follows:

$$\begin{aligned} & \max \quad \sum_{r=1}^{s} u_r y_{ro} \\ & S.t \quad \sum_{i=1}^{m} v_i x_{io} = 1 \qquad (1) \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \dots, n \\ & u_r \geq \varepsilon, \quad r = 1, \dots, s \quad v_i \geq \varepsilon, \quad i = 1\dots, m \end{aligned}$$

By adding the constrains $1\lambda = 1$ the BCC model [1], obtained respectively.

In continue DEA models without any inputs or any outputs are discussed. Also, DEA models with fixed inputs or fixed outputs are presented. We will show that the CCR model without any inputs or any outputs is meaningless; and, the CCR model with fixed inputs or fixed outputs is equivalent to the BCC model without any inputs respectively. Then, we will show that the BCC model with fixed inputs or fixed outputs can be considered as a BCC model without any inputs or any outputs. Finally, it will prove that BCC model without any inputs or any outputs can be transformed to a model with fewer variables and constraints.

Theorem 1

The CCR model without any inputs is meaningless in output oriented form.

Proof. Consider the following CCR model without any inputs to measure the efficiency score of DMU_o:

$$\begin{aligned} & \max \ \varphi \\ & S.t \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro}, \qquad r = 1, \dots, s \qquad (2) \\ & \lambda_j \geq 0, \qquad j = 1, \dots, n \end{aligned}$$

It is clear that $\lambda_o = 1$ and $\lambda_j = 0,\ j = 1, ..., n, j \neq o$ is a feasible solution of model 2, moreover $\lambda_o = k$ and $\lambda_j = 0,\ j = 1, ..., n, j \neq o$ is also a feasible solution of this model. It means that when $k \to +\infty$ the optimal value of the objective function tents to infinity: $\varphi^* \to +\infty$ and clearly the evaluated unit is infinitely inefficient. This is completed the proof. Therefore, the CCR model without any inputs is meaningless.

Theorem 2

The CCR model without any outputs is meaningless in input oriented form.

Proof. Consider the following CCR model without any outputs to measure the efficiency score of DMU_o:

$$\begin{aligned} & \min \ \theta \\ & S.t \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \qquad i = 1, \dots, m \qquad (3) \\ & \lambda_j \geq 0, \qquad j = 1, \dots, n \end{aligned}$$

It is clear that $\lambda_j \geq 0,\ j = 1, ..., n$ and $\theta^* < 0$ is not a feasible solution of model 3, then $\theta^* = 0$. In other words all units evaluated with zero score, i.e. all units are evaluated absolutely inefficient. Therefore, the CCR model without any outputs is meaningless.

Theorem 3

A CCR model with a fixed input in output oriented form is equivalent to a BCC model without any input.

Proof. Consider the following CCR model

with a non zero fixed input a in output oriented development form:

$$\max \ \varphi$$
$$S.t \quad \sum_{j=1}^{n}\lambda_j a \le a,$$
$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi y_{ro}, \qquad r=1,\dots,s \tag{1}$$
$$\lambda_j \ge 0, \qquad j=1,\dots,n$$

We want to show that if (φ^*,λ^*) be the optimal solution of 4.2.3 then $\sum_{j=1}^{n}\lambda_j^* = 1$. By contradiction, suppose that $\sum_{j=1}^{n}\lambda_j^* < 1$. Let $\overline{\lambda_j} = \frac{\lambda_j^*}{\sum_{j=1}^{n}\lambda_j^*}$, then we get:

$$\sum_{j=1}^{n}\overline{\lambda_j} y_{rj} = \sum_{j=1}^{n}\frac{\lambda_j^*}{\sum_{j=1}^{n}\lambda_j^*} y_{rj} \ge$$
$$\frac{\varphi^* y_{ro}}{\sum_{j=1}^{n}\lambda_j^*} = \frac{\varphi^*}{\sum_{j=1}^{n}\lambda_j^*} y_{ro} > \varphi^* y_{to}, \quad r=1,\dots,s \tag{2}$$

Therefore, $\overline{\varphi} = \frac{\varphi^*}{\sum_{j=1}^{n}\lambda_j^*} > \varphi^*$ and it is contradicting with the optimality of φ^*; therefore, $\sum_{j=1}^{n}\lambda_j^* = 1$. Then the following BCC model is obtained:

$$\max \ \varphi$$
$$S.t \quad \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi y_{ro}, \qquad r=1,\dots,s$$
$$\sum_{j=1}^{n}\lambda_j = 1 \tag{3}$$
$$\lambda_j \ge 0, \qquad j=1,\dots,n$$

Theorem 4

The CCR model with fixed outputs in input oriented form is equivalent to a BCC model without any output.

Proof is similar to theorem 3.

Theorem 5

The BCC model with fixed inputs in output oriented form is equivalent to a BCC model without any inputs.

Proof. Consider the following mentioned BCC model:

$$\max \ \varphi$$
$$S.t \quad \sum_{j=1}^{n}\lambda_j a \le a,$$
$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi y_{ro}, \qquad r=1,\dots,s$$
$$\sum_{j=1}^{n}\lambda_j = 1 \tag{4}$$
$$\lambda_j \ge 0, \qquad j=1,\dots,n$$

Clearly the following model can be considered:

$$\max \ \varphi$$
$$S.t \quad \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi y_{ro}, \qquad r=1,\dots,s$$
$$\sum_{j=1}^{n}\lambda_j = 1 \tag{5}$$
$$\lambda_j \ge 0, \qquad j=1,\dots,n$$

And the proof is completed.

Now, we are going to present some DEA models to generate the kocal weight from a comparison pairwise matrix. The development output oriented form of BCC model and DEA/MOLP models are used in this way.

3. DEA/AHP Models to generate the favorable weights

Again consider the following pairwise comparison matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{6}$$

Where $a_{ii}=1, i=1,\dots,n$, $a_{ij}=\frac{1}{a_{ji}}, i \ne j$

and $W=(w_1,...,w_n)$ be its weights vector. DEA/AHP views each row of the matrix A as a decision making entity, which is referred to as decision making unit in DEA, and each column as an output. A dummy input that has a value of 1 is assumed for all the DMUs. Each DMU has in fact n outputs and one dummy constant input, based on which the following input-oriented CCR model is built by Ramanathan [8] to estimate the local weights of the comparison matrix A :

$$\begin{aligned} &\max\ w_{i_o}=\sum_{j=1}^{n}u_j a_{i_o j} \\ &S.t\ \ v=1 \\ &\sum_{j=1}^{n}u_j a_{ij}-v\le 0,\ i=1,...,n \\ &u_j,v\ge 0,\qquad j=1,\dots,n \end{aligned} \tag{7}$$

Where $i_o\in\{1,...,n\}$ represents the DMU under evaluation, that is DMU_o. The optimum objective function value of the above model, w_{i_o} , represents the DEA efficiency of DMU_o and is used as its local weight. The LP model 4.3.2 is solved for all the DMUs to obtain the local weight vector W of the comparison matrix A. It has been proved [6] that DEA/AHP can estimate the true weights if A is a perfectly consistent comparison matrix.

Now, consider the following output oriented CCR model with a fixed input:

$$\begin{aligned} &\max\ \varphi \\ &S.t\quad \sum_{i=1}^{n}\lambda_i a\le a, \\ &\sum_{i=1}^{n}\lambda_i a_{ij}\ge \varphi a_{i_o j},\qquad j=1,\dots,n \\ &\lambda_i\ge 0,\qquad i=1,\dots,n \end{aligned} \tag{8}$$

Where $i_o\in\{1,...,n\}$ represents the DMU under evaluation, that is DMU_o. Follow to theorem 4.2.3 this model is equivalent to the BCC model without any input in output oriented development form and we get the following model to generate the favorable weights from a pairwise comparison matrix:

$$\begin{aligned} &\max\ \varphi \\ &S.t\ \ \sum_{i=1}^{n}\lambda_i a_{ij}\ge \varphi a_{i_o j},\qquad j=1,\dots,n \\ &\sum_{i=1}^{n}\lambda_i=1 \\ &\lambda_i\ge 0,\qquad i=1,\dots,n \end{aligned} \tag{9}$$

We proposed model 12 as a DEA/AHP model to generate the local weights. Clearly, n linear programming problems with $n+1$ variables and $n+1$ constraints should be solved to find the weights. The optimal objective value of model 4.3.4 can use as the local weights of alternatives and criterion. Obviously, the summation of local weights are equal to unity in AHP models therefore, we can use the normalized weights as the local weights (dividing each obtained weight to summation of all generated weights). The final weights of each alternative can calculate by the following formula:

$$\textit{Final weight of } A_i = \sum_j [\begin{pmatrix}\textit{Local weight of } A_i \textit{ with}\\ \textit{respect of criterion } C_j\end{pmatrix}\times \begin{pmatrix}\textit{Local weight of}\\ \textit{criterion } C_j\end{pmatrix}] \tag{10}$$

For all i.

Again consider the matrix 9. As mentioned in this section we considered the efficiency of each alternatives (and criterion) as the local weights of pairwise comparison matrix. Then, consider the following MOLP/DEA model to obtain the weights:

$$\max\ \min\{\sum_{i=1}^{n}u_i a_{i1},...,\sum_{i=1}^{n}u_i a_{in}\}$$

$$S.t\ \sum_{i=1}^{n} u_i a_{ij} \le 1, \qquad j=1,\dots,n$$
$$u_i \ge 0, \qquad i=1,\dots,n \tag{11}$$

The main idea of this model is that we want to maximize the minimum of the weights. After linearizing, the following linear programming problem is obtained:

$$\max\ \varphi$$
$$S.t\ \sum_{i=1}^{n} u_i a_{ij} \le 1, \qquad j=1,\dots,n$$
$$\sum_{i=1}^{n} u_i a_{ij} - \varphi \ge 0 \tag{12}$$
$$u_i \ge 0, \qquad i=1,\dots,n$$

It is clear that only one problem should be solved to generate the weight. One may say that because of using MOLP technique an infinite number of optimal solutions may be at hand, therefore, the obtained weights (efficiency scores) may be changed. To answer this problem it is necessary to note that the weights are not unique in original AHP methods and they may change by using different kinds of methods. Finally, the weights are calculated as :

$w_i = \sum_{i=1}^{n} u_i^* a_{ij}$, $i=1,\dots,n$, we can normalize the weights to satisfy the AHP properties.

Maximizing the weights of alternatives and criterion is another way to find the weights. The proposed "common set of weights" model is as follows:

$$\max\ \{\sum_{i=1}^{n} u_i a_{i1}, \dots, \sum_{i=1}^{n} u_i a_{in}\}$$
$$S.t\ \sum_{i=1}^{n} u_i a_{ij} \le 1, \qquad j=1,\dots,n \tag{13}$$
$$u_i \ge 0, \qquad i=1,\dots,n$$

This model transformed to the following linear programming problem using goal programming techniques as follows:

$$\min \sum_{i=1}^{n} z_i$$
$$S.t\ \sum_{i=1}^{n} u_i a_{ij} + z_i = 1, \qquad j=1,\dots,n \tag{14}$$
$$u_i, z_i \ge 0, \qquad i=1,\dots,n$$

It should be noted that only one problem is used to generate the weights from models 15 and 17, therefore, the computational process is so decreased.

4. Numerical Example

In this section we are going to add our proposals in a numerical example and compare them with the results of AHP method and model 4.3.2. In this way, consider a hierarchy process with three alternatives $A_1, A_2\ and\ A_3$ and four criterion $C_1, C_2, C_3\ and\ C_4$.

The pairwise comparison matrices

Table 1: Comparison of criterion with respect to goal

DMUs	C_1	C_2	C_3	C_4
C_1	1	1	4	5
C_2	1	1	5	3
C_3	1/4	1/5	1	3
C_4	1/5	1/3	1/3	1

Table 2: Comparison of alternatives with respect to C_1

DMUs	A_1	A_2	A_3
A_1	1	1/3	5
A_2	3	1	7
A_3	1/5	1/7	1

Table 3: Comparison of alternatives with respect to C_2

DMUs	A_1	A_2	A_3
A_1	1	1/9	1/5
A_2	9	1	4
A_3	5	1/4	1

Table 4: Comparison of alternatives with respect to C_3

DMUs	A_1	A_2	A_3
A_1	1	2	5
A_2	1/2	1	3
A_3	1/5	1/3	1

Table 5: Comparison of alternatives with respect to C_4

DMUs	A_1	A_2	A_3
A_1	1	3	9
A_2	1/3	1	3
A_3	1/9	1/3	1

Following tables show the generated weights by AHP method, model 4.3.2, BCC formulation (model 4.3.4), model 4.3.7 and finally, model 4.3.9:

Table 6: Generated weights by AHP method

	C_1	C_2	C_3	C_4	Final Weights
A_1	0.279	0.060	0.582	0.692	0.261
A_2	0.649	0.709	0.309	0.231	0.590
A_3	0.072	0.231	0.109	0.077	0.148
	0.400	0.394	0.128	0.078	

Table 7: Generated weights by DEA/AHP model 10

	C_1	C_2	C_3	C_4	Final Weights
A_1	0.714	0.111	1	1	0.712
A_2	1	1	0.600	0.333	1
A_3	0.143	0.556	0.200	0.111	0.346

Table 8: Generated weights by BCC model

	C_1	C_2	C_3	C_4	Final Weights
A_1	0.385	0.067	0.556	0.692	0.346
A_2	0.538	0.600	0.333	0.231	0.483
A_3	0.077	0.333	0.111	0.077	0.171
	0.341	0.341	0.205	0.113	

Table 9: Generated weights by model 15

	C_1	C_2	C_3	C_4	Final Weights
A_1	0.226	0.067	0.714	0.692	0.283
A_2	0.677	0.600	0.143	0.231	0.532
A_3	0.097	0.333	0.143	0.077	0.185
	0.400	0.360	0.120	0.120	

Table 10: Generated weights by model 17

	C_1	C_2	C_3	C_4	Final Weights
A_1	0.385	0.067	0.556	0.692	0.313
A_2	0.538	0.600	0.333	0.231	0.506
A_3	0.077	0.333	0.121	0.077	0.181
	0.400	0.360	0.120	0.120	

note that the rank of alternatives remains unchanged in all the mentioned models. But there are some comparison between these models.

5. Conclusion

In this paper some DEA/AHP models presented to find favorable local weights from a comparison matrix. the following results obtained:

- Consider the DEA/AHP model 10 n linear programming problems which each of them has $n+1$ constraints and $n+1$ variables should be solved to generate the weights.
- The BCC formulation (model 12) should be solved n times to generate the weights which each of them has $n+1$ constraints and $n+1$ variables.
- Only one problem need to be solved by using models 15 and 17 (MOLP/DEA models) which these models has only n constraints and n variables.

Inconsistency did not discuss in this paper and may be considered for feature studies

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