Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 1, 2022 Article ID IJIM-1474, 13 pages
DOI: http://doi.org/10.30495/ijim.2022.19219
Research Article

Science and Research Branch (IAU)

On the Independence of Jeffreys' Prior for Truncated-Exponential Skew-Symmetric Models

S. Mirzadeh *, A. Iranmanesh †‡, E. Ormoz §

Received Date: 2020-11-19 Revised Date: 2021-02-14 Accepted Date: 2021-10-02

Abstract

We study the Jeffreys' prior of the skewness parameter of truncated-exponential skew-symmetric distributions (TESSDs). We show that this prior is symmetric, improper, and with tails $O(|\lambda|^{-1})$. While Jeffreys' prior is improper, as we have shown, it yields a proper posterior distribution for some densities. We also calculate the independent Jeffreys' prior for the case of unknown location and scale parameters and show that the corresponding posterior distribution is proper. A simulation study using Monte Carlo methods is presented to compare the efficiency of Bayesian estimators in TESSD with Azzalinis' skew models by computing the bias and the mean square error under square error loss and Linex loss functions. The results show the superiority of the Bayesian estimators in TESSD versus Bayesian estimators in Azzalinis' skew models.

Keywords : Bayesian estimator; Jeffreys' prior; Posterior existence; Simulation; Truncated-exponential skew-symmetric distributions; Truncated exponential skew-logistic distributions.

1 Introduction

Nowadays skew distributions are required to model asymmetric data. Such data frequently arise in many domains such as biometry, finance, materials sciences, and environmetrics. See for instance Ley [14] for detailed explanations. A popular method to produce this kind of distribution is based on adding a skew parameter to a symmetric distribution that controls skewness. In this line, univariate skew-symmetric models have been introduced by many authors. Azzalini [1, 2] proposed an asymmetric density, as follows

$$f_Y(y) = 2f_X(y)F_X(\lambda y), \quad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}. \tag{1.1}$$

A special case of (1.1) is the class of skew-normal distributions, which is given by

$$f_Y(y) = \frac{2}{\sigma}\varphi\left(\frac{y-\mu}{\sigma}\right)\phi\left(\lambda\frac{y-\mu}{\sigma}\right),$$
$$y \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0,\ \lambda \in \mathbb{R}, \tag{1.2}$$

where φ and ϕ are the standard normal probability density function (PDF) and cumulative distribution function (CDF), respectively. It is denoted by $Y \sim SN(\mu, \sigma, \lambda)$. Some other special

*Department of Mathematics and Statistics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
†Corresponding author. Iranmanesh0030@mshdiau.ac.ir, Tel:+98(915)3101271.
‡Department of Mathematics and Statistics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
§Department of Mathematics and Statistics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

cases of (1.1) are skew-t, skew-logistic, and skew-Cauchy distributions. Wang et al. [23] showed that this method can be extended to any continuous symmetric density f, with support on $\mathbb{R}$ and mode at 0, through the transformation as follows:

$$h_Y(y) = \frac{2}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right) \pi\left(\lambda \frac{y-\mu}{\sigma}\right),$$
$$y \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0,\ \lambda \in \mathbb{R}, \qquad (1.3)$$

where π, named the skewing function, is a function that satisfies $0 \leq \pi(y) \leq 1$ and $\pi(-y) = 1 - \pi(y)$. It follows that any symmetric CDF can be used as a skewing function. Several functions for f and π have been used in the literature; see Azzalini and Capitanio [4] for the skew Student-t distribution, Nadarajah [19] for the skew-logistic distribution and Azzalini [2] for the power exponential distribution.

Subsequently, Ferreira and Steel [10] defined Y as a new skew version of random variables with the PDF as follows:

$$f_Y(y) = f_X(y) w(F_X(y)), \quad y \in \mathbb{R}, \qquad (1.4)$$

where $w(\cdot)$ is a PDF on $(0,1)$ and X is a symmetric random variable about zero with PDF $f_X(\cdot)$ and CDF $F_X(\cdot)$. By changing $w(\cdot)$ in (1.4), many of the known families of skew distributions are obtained. For example, if $w(x) = 2F_X(\lambda F_X^{-1}(x))$, then (1.4) changes to (1.1).

Distributions obtained using this method are called skew-symmetric distributions. For a recent and extensive overview, we refer the reader to Jones [13].

The Azzalini-type distributions have garnered much success due to their strong stochastic properties, elegant generating mechanisms, and good fitting properties, (Azzalini and Genton [5]). Unfortunately, some certain Azzalini-type distributions suffer from inferential problems. Ley and Paindaveine [15] and Hallin and Ley [12] determined that, among others, the skew-normal distribution has singular Fisher information scores due to the collinearity of the location and skewness scores. Pewsey [21] proved that the maximum likelihood estimation for the skewness parameter in Azzalini skew-normal families does not always exist. Also, Azzalini [1] showed that the Fisher information matrix of the parameters (μ, σ, λ) is singular at $\lambda = 0$ for the skew-normal sampling model. Furthermore, in some data sets the maximum likelihood estimate of the skewness parameter tends toward infinity. That is noteworthy that a half-normal distribution is fitted when values are observed on both halves of the distribution(Azzalini and Capitanio [3].

Some authors have proposed the use of the Bayesian approach to avoid to avoid these inferential problems. Although λ is named skewness parameter, this parameter controls the mode, asymmetry, spread, and tail behavior of the PDF. Despite this, a lot of priors for λ have been proposed in the literature, inter alia by Liseo and Loperfido [16, 17], Cabras et al. [7], Branco et al. [6], and Rubio and Liseo [22]. These references focus on the construction of noninformative priors from different viewpoints. For example, Rubio and Liseo [22] studied the Jeffreys' prior of the skewness parameter of a general class of scalar skew-symmetric models. They also calculated the independent Jeffreys' prior for the case of unknown location and scale parameters and investigated conditions for the propriety of the corresponding posterior distribution.

Recently Dette et al. [9] have studied priors based on distances for skew-symmetric models and interpreted the parameter λ as the perturbation parameter. They proposed a new method for constructing priors for λ based on its overall effect on the shape of the density. For this purpose, they defined a measure of perturbation and based on that, built informative priors. Canale et al. [8], have also studied informative priors and proposed the use of normal and skew-normal priors for λ in the skew-normal model. Another recent review can be found in Ghaderinezhad et al. [11].

Nadarajah et al. [20] introduced a new family of skew distributions as a competitor to the well-known Azzalini skew distributions, called truncated-exponential skew-symmetric distributions (TESSDs). They showed that TESSD is a member of the exponential family; therefore, the estimate of the skewness parameter can be obtained easier.

A random variable Y has the truncated-exponential skew-symmetric (TESS) distribution, denoted by $Y \sim TESS(\mu, \sigma, \lambda)$, if its PDF is given by

$$f_Y(y;\mu,\sigma,\lambda) = \frac{\lambda}{\sigma[1-\exp(-\lambda)]} f_X\left(\frac{y-\mu}{\sigma}\right) \times \exp\left[-\lambda F_X\left(\frac{y-\mu}{\sigma}\right)\right],$$
$$y, \mu, \lambda \in \mathbb{R},\ \sigma > 0, \qquad (1.5)$$

where $f_X(\cdot)$ and $F_X(\cdot)$ are the PDF and CDF of a symmetric random variable X about zero, respectively, and λ is a shape parameter. If $\mu = 0$

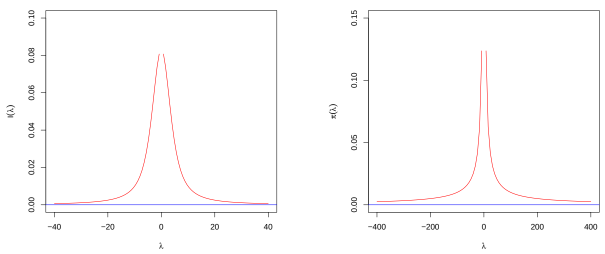

Figure 1: Plots of $I(\lambda)$ and $\pi(\lambda)$.

and $\sigma = 1$, then it is denoted by $Y \sim TESS(\lambda)$. Also note that (1.5) is a particular case of (1.4) for $w(x) = \frac{\lambda \exp(\lambda x)}{1-\exp(\lambda)}$. Also (1.4) is undefined at $\lambda = 0$, so $\lambda = 0$ should be interpreted as the $\lim \lambda \to 0$. If $\lambda \to \pm\infty$, then $Y \sim TESS(\lambda)$ reduces to degenerate random variables. Note also that (1.5) is symmetric concerning λ in the sense that $f(y;\lambda) = f(-y;-\lambda)$. Furthermore, in the limit, as $\lambda \to 0$, $Y \sim TESS(\lambda)$ has the same distribution as X.
The content of the rest of this paper is organized as follows. In Section 2, we study the Jeffreys' prior of the skewness parameter of $TESS$ models. We show that this prior is symmetric, improper, and with tails $O(|\lambda|^{-1})$. We also calculate the independence Jeffreys' prior for the case of unknown location and scale parameters and show that the corresponding posterior distribution is proper. Section 3 provides Bayesian inference for the truncated exponential skew-logistic distribution. A simulation study is conducted in Section 4 to compare the performances of the Bayesian estimators in TESSD with Azzalini's skew models. Finally, some conclusions including the advantages of the proposed method are noted in Section 5.

2 Independence Jeffreys prior for TESS(μ, σ, λ) models

First, consider the particular skew-symmetric model (1.5) without location and scale parameters, that is, suppose that $\mu = 0$ and $\sigma = 1$. Let $Y \sim TESS(\lambda)$; then the PDF and CDF of Y are,

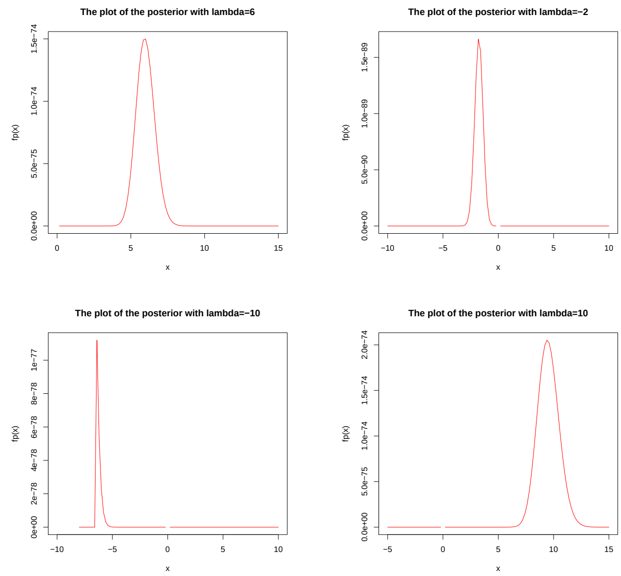

Figure 2: The Plot of $\pi(\lambda|\boldsymbol{y})$ in (3.11) for different values of λ.

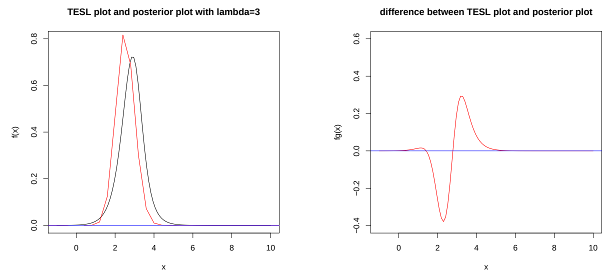

Figure 3: $\pi(\lambda|\boldsymbol{y})$ in (3.11) for $\lambda = 3$ and $TESL(3)$ (left), the difference between the $\pi(\lambda|\boldsymbol{y})$ and $TESL(3)$ (right).

Table 1: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\lambda = 2)$

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	1.98424	−0.01576	0.17300	1.98055	−0.01945	0.06207
	200	1.96046	−0.03954	0.08796	1.99859	−0.00142	0.04103
	500	2.02301	0.02301	0.02685	1.99905	−0.00095	0.01831
	1000	1.98806	−0.01194	0.01282	2.00526	0.00526	0.01067
LINEX,c=0.5	100	1.94021	−0.05979	0.15894	1.96254	−0.03746	0.06269
	200	1.93951	−0.06049	0.08572	1.98953	−0.01047	0.04101
	500	2.01428	0.01428	0.02605	1.99387	−0.00613	0.01831
	1000	1.98386	−0.01614	0.01283	2.00285	0.00285	0.01064
LINEX,c=2	100	1.82459	−0.17541	0.14898	1.90892	−0.09108	0.06841
	200	1.88053	−0.11947	0.08537	1.96249	−0.03751	0.04192
	500	1.98874	−0.01126	0.02462	1.97836	−0.02164	0.01863
	1000	1.97140	−0.02860	0.01305	1.99561	−0.00439	0.01062

Table 2: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\lambda = 5$)

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	5.37587	0.37587	3.52230	5.02328	0.02328	0.09566
	200	4.98018	−0.01980	0.82275	4.99672	−0.00328	0.05815
	500	4.99606	−0.00394	0.26541	4.97785	−0.02215	0.02582
	1000	5.03436	0.03436	0.14907	4.99261	−0.00740	0.01711
LINEX,c=0.5	100	4.80823	−0.19177	1.36594	4.99799	−0.00201	0.09380
	200	4.76423	−0.23578	0.68166	4.98169	−0.01831	0.05798
	500	4.91012	−0.08988	0.24884	4.96952	−0.03048	0.02615
	1000	4.99048	−0.00952	0.14102	4.98760	−0.01240	0.01716
LINEX,c=2	100	3.95358	−1.04642	1.55001	4.92348	−0.07652	0.09585
	200	4.28313	−0.71687	0.86415	4.93706	0.06294	0.06018
	500	4.68212	−0.31788	0.28718	4.94467	−0.05533	0.02795
	1000	4.86699	−0.13301	0.14074	4.97262	−0.02739	0.01761

Table 3: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\lambda = 10$)

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	13.72096	3.72096	289.3451	10.28554	0.28554	1.30000
	200	10.83872	0.83872	10.84226	9.96891	−0.03109	0.53306
	500	10.29852	0.29852	4.62252	10.10369	0.10369	0.22020
	1000	10.00312	0.00312	1.50353	10.01389	0.01389	0.12617
LINEX,c=0.5	100	8.14897	−1.85103	7.56273	10.02631	0.02631	1.09922
	200	9.10914	−0.89086	4.29390	9.84539	−0.15615	0.53093
	500	9.61212	−0.38882	2.87193	10.05268	0.05268	0.20802
	1000	9.69159	−0.30841	1.32848	9.98875	−0.01125	0.12484
LINEX,c=2	100	5.69502	−4.30498	19.3300	9.33902	−0.66098	1.26784
	200	7.00555	−2.99445	9.92526	9.49797	−0.50203	0.69404
	500	8.31978	−1.68022	3.94971	9.90349	−0.09651	0.20258
	1000	8.94788	−1.05212	1.88404	9.91425	−0.08575	0.12840

Table 4: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\lambda = 20$)

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	42.81583	22.81583	3448.152	20.19108	0.19108	4.96704
	200	25.64084	5.64084	408.2569	19.96143	−0.03857	2.26955
	500	21.52976	1.52976	35.98669	19.93658	−0.06342	0.72617
	1000	20.25917	0.25917	11.32517	20.04948	0.04948	0.52799
LINEX,c=0.5	100	11.52483	−8.47517	77.35188	19.22393	−0.77607	4.63847
	200	14.29944	−5.70056	38.35158	19.47654	−0.52346	2.32437
	500	17.08368	−2.91632	16.22780	19.74015	0.25985	0.76145
	1000	18.08021	−1.91979	9.38291	19.94973	−0.05027	0.51765
LINEX,c=2	100	6.97231	−13.0277	170.4935	16.96215	−3.03785	11.774155
	200	9.32772	−10.67228	114.9091	18.20015	−1.79985	4.79719
	500	12.63988	−7.36017	56.16614	19.18056	−0.81944	1.28860
	1000	14.76564	−5.23436	29.53819	19.65813	−0.34187	0.60245

Table 5: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\lambda = -5$)

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	−5.58600	−0.58600	4.15865	−5.01806	−0.01806	0.30302
	200	−5.10090	−0.10090	0.95814	−5.05716	−0.05716	0.143085
	500	−5.14004	−0.14004	0.37036	−5.01198	−0.01198	0.05926
	1000	−5.02944	−0.02944	0.20073	−5.00034	−0.00034	0.024171
LINEX,c=0.5	100	−7.49926	−2.49926	58.1051	−5.09604	−0.09604	0.32612
	200	−5.38883	−0.38883	1.49174	−5.09584	−0.09584	0.15194
	500	−5.24035	−0.24035	0.44966	−5.02720	−0.02720	0.06035
	1000	−5.075148	−0.07514	0.21632	−5.00790	−0.00790	0.02433
LINEX,c=2	100	−11.61105	−6.61105	138.3776	−5.34495	−0.34495	0.48654
	200	−6.706633	−1.70663	7.20022	−5.21530	−.21535	0.19881
	500	−5.59263	−0.59763	0.94077	−5.07339	−0.07339	0.06656
	1000	−5.22288	−0.22288	0.29954	−5.03073	−0.03073	0.02552

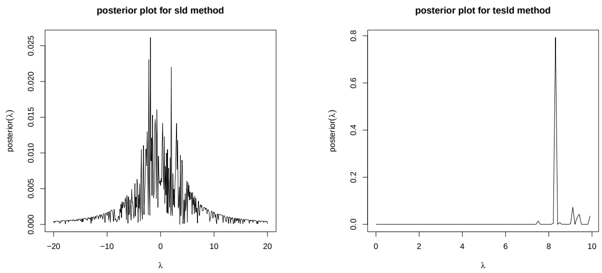

Figure 4: Posterior plot for $ASLD$, $TESLD$.

Table 6: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\lambda = -10$)

		SLD			$TESLD$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	−15.89797	−5.89797	366.9605	−10.00481	−0.00481	0.79240
	200	−11.16210	−1.16210	24.43663	−9.94691	0.05309	0.47324
	500	−10.71573	−0.71573	5.42981	−10.00853	−0.00853	0.17898
	1000	−10.17523	−0.17523	1.18961	−10.00294	−0.00294	0.13815
LINEX,c=0.5	100	−21.32145	−11.32145	84.56781	−10.26717	−0.26717	0.94820
	200	−10.03345	−8.03345	42.78654	−10.07409	−0.07409	0.50002
	500	−10.90889	−1.90889	15.46144	−10.05929	−0.05929	0.18695
	1000	−10.54683	−0.54683	1.75476	−10.02820	−0.02820	0.14036
LINEX,c=2	100	−32.08761	−21.08761	1322.675	−11.16717	−1.16717	2.57336
	200	−24.14672	−16.14672	721.754	−10.48419	−0.48419	0.81351
	500	−15.89221	−5.89221	66.45959	−10.21575	−0.21575	0.24146
	1000	−12.03512	−2.03512	7.30468	−10.10507	−0.10507	0.15501

respectively, given by

$$f_Y(y) = \frac{\lambda}{1-\exp(-\lambda)} f_X(y) \exp\left\{-\lambda F_X(y)\right\},$$

$$F_Y(y) = \frac{1-\exp\{-\lambda F_X(y)\}}{1-\exp(-\lambda)}, \quad y, \lambda \in \mathbb{R}. \qquad (2.6)$$

Recall that the Fisher's information of the parameter λ is defined as

$$\begin{aligned} I(\lambda) &= E\left[\frac{\partial}{\partial\lambda}\log\left(f_Y(Y)\right)\right]^2 \\ &= E\left[\left(\frac{1}{\lambda} - \frac{\exp(-\lambda)}{1-\exp(-\lambda)}\right)^2\right. \\ &- 2\left(\frac{1}{\lambda} - \frac{\exp(-\lambda)}{1-\exp(-\lambda)}\right) F_X(Y) + F_X^2(Y)] \end{aligned}$$

where

$$E\left(F_X(Y)\right)$$

Table 7: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, ($\lambda = 2$)

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	2.02319	0.02319	0.20848	1.96992	−0.03008	0.14885
	200	1.97681	−0.02320	0.06398	2.01111	0.01111	0.07122
	500	2.02700	0.02700	0.02780	2.01747	0.01747	0.02400
	1000	2.00753	0.00753	0.01610	2.00987	0.00987	0.01285
LINEX,c=0.5	100	1.98528	−0.01472	0.18625	1.93377	−0.06623	0.15019
	200	1.95989	−0.04011	0.06250	1.99286	−0.00714	0.07066
	500	2.01998	0.01998	0.02699	2.01229	0.01229	0.02381
	1000	2.00412	0.00412	0.01593	2.00625	0.00625	0.01277
LINEX,c=2	100	1.88328	−0.11673	0.15347	1.82701	−0.17300	0.17067
	200	1.91132	−0.08868	0.06198	1.93862	−0.06139	0.07300
	500	1.99928	−0.00072	0.02523	1.99678	−0.00322	0.02354
	1000	1.99398	−0.00602	0.01560	1.99540	−0.00460	0.01270

Table 8: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, $(\lambda = 5)$

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	5.48968	0.48968	5.15011	5.06612	0.06612	0.29335
	200	5.18656	0.18656	0.90906	5.02383	0.02383	0.16488
	500	5.05139	0.05139	0.36635	4.99310	−0.00069	0.06002
	1000	5.00184	0.00184	0.15494	4.97858	−0.02142	0.02562
LINEX,c=0.5	100	4.90583	−0.09418	1.54039	4.98951	−0.01049	0.27730
	200	4.97054	−0.02947	0.68097	4.98581	−0.01419	0.16108
	500	4.97280	−0.02720	0.33319	4.98428	−0.01572	0.05978
	1000	4.96355	−0.03645	0.14391	4.97109	−0.02891	0.02589
LINEX,c=2	100	4.04705	−0.95295	1.34876	4.77153	−0.22847	0.29913
	200	4.47513	−0.52477	0.65661	4.87489	−0.12511	0.16697
	500	4.75954	−0.24045	0.31796	4.93967	−0.06033	0.06175
	1000	4.85426	−0.14574	0.15098	4.94873	−0.05127	0.02730

Table 9: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, $(\lambda = 10)$

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	22.46437	12.46437	2080.787	10.09783	0.09783	0.93225
	200	14.52433	4.52433	699.6167	10.08433	0.08433	0.62562
	500	10.14684	0.14684	2.28785	10.03488	0.03488	0.19822
	1000	9.97799	−0.02001	0.98952	10.03090	0.03090	0.09793
LINEX,c=0.5	100	8.77100	−1.22900	8.03554	9.84751	−0.15429	0.085643
	200	9.42280	−0.57730	6.51414	9.95785	−0.04215	0.58561
	500	9.55803	−0.44197	1.80811	9.98446	−0.01553	0.01933
	1000	9.69456	−0.30544	0.92517	10.00568	0.00568	0.09606
LINEX,c=2	100	5.93704	−4.06296	17.44998	9.18279	−0.18271	1.30309
	200	7.18702	−2.81298	9.21855	9.60191	−0.39810	0.66576
	500	8.32640	−1.67360	3.57127	9.83685	−0.16315	0.28540
	1000	8.981458	−1.01855	1.57138	9.93097	−0.06904	0.09804

Table 10: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, $(\lambda = 20)$

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	64.29655	44.29655	7389.672	20.53110	0.53110	4.65254
	200	40.42953	20.42953	3512.522	19.96831	−0.03168	1.85143
	500	22.34251	2.34251	69.79996	19.98528	−0.01472	0.79206
	1000	21.08524	1.08524	20.49246	20.00892	0.00892	0.32721
LINEX,c=0.5	100	12.65343	−7.34657	62.55216	19.53724	−0.46276	3.78772
	200	14.81486	−5.18514	37.91769	19.48399	−0.51601	1.94955
	500	17.51208	−2.48792	15.92410	19.78747	−0.21523	0.80583
	1000	18.68688	−1.31312	10.51897	19.90949	−0.09051	0.32876
LINEX,c=2	100	7.32701	−12.67299	161.6213	17.24412	−2.75588	9.89251
	200	9.44080	−10.55920	113.2952	18.21297	−1.78703	4.49488
	500	12.83165	−7.16835	54.03411	19.22507	−0.77493	1.28137
	1000	15.09597	−4.90404	27.19161	19.61958	−0.38042	0.44637

Table 11: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, $(\lambda = -5)$

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	−5.46346	−0.46399	2.85308	−5.05773	−0.05773	0.33506
	200	−5.48392	−0.48392	2.31935	−5.08719	−0.08719	0.18176
	500	−5.05585	−0.05585	0.31429	−5.00740	−0.00740	0.05903
	1000	−5.00070	−0.00070	0.11211	−5.03328	−0.03328	0.02544
LINEX,c=0.5	100	−6.38606	−1.38606	10.18978	−5.13591	−0.13591	0.36529
	200	−5.86281	−0.86281	4.69724	−5.12634	−0.12634	0.19378
	500	−5.17158	−0.17158	0.36912	−5.02258	−0.02258	0.05999
	1000	−5.03991	−0.03991	0.11917	−5.04093	−0.04093	0.02610
LINEX,c=2	100	−9.42403	−4.42403	48.55797	−5.38672	−0.38672	0.55103
	200	−7.23875	−2.23875	17.35753	−5.24738	−.24738	0.25090
	500	−5.46069	−0.46069	0.69135	−5.06866	−0.06866	0.06574
	1000	−5.16399	−0.16399	0.16348	−5.06400	−0.06400	0.02883

Table 12: Comparison of Bayesian estimators for the $TESND$ versus $ASND$, $(\lambda = -10)$

		SND			$TESND$		
	n	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$	λ_b	$Ab(\lambda_b)$	$MSE(\lambda_b)$
SEL	100	−24.36395	−14.36395	1859.234	−9.97350	0.02650	0.84848
	200	−11.84527	−1.84527	35.87773	−10.02179	−0.02179	0.52897
	500	−10.42931	−0.42931	3.34711	−10.09797	−0.09797	0.26604
	1000	−10.13880	−0.13880	1.41036	−9.98897	0.01103	0.09639
LINEX,c=0.5	100	−28.04521	−18.04521	2201.103	−10.23485	−0.23485	0.99680
	200	−18.74903	−8.74903	452.103	−10.15050	−0.15050	0.57846
	500	−11.24279	−1.24279	6.86195	−10.14945	−0.14945	0.28384
	1000	−10.47716	−0.47716	1.95325	−10.01412	−0.01412	0.09742
LINEX,c=2	100	−30.41805	−20.41805	2431.946	−11.14108	−1.14108	2.61885
	200	−25.99065	−15.99065	741.2946	−10.56462	−0.56462	0.97304
	500	−14.07392	−4.07392	30.13288	−10.30802	−0.30802	0.37236
	1000	−11.70974	−1.70974	6.30170	−10.09059	−0.09059	0.10839

$$= \int_{-\infty}^{\infty} F_X(y) \frac{\lambda}{1-\exp(-\lambda)} f_X(y)$$

$$\times \exp(-\lambda F_X(y)) dy$$

$$= \frac{1}{1-\exp(-\lambda)} \left(-\exp(-\lambda) - \frac{1}{\lambda}\exp(-\lambda) + \frac{1}{\lambda} \right)$$

and

$$E\left(F_X^2(Y)\right) = \frac{1}{1-\exp(-\lambda)}$$

$$\left(-\exp(-\lambda) - \frac{2}{\lambda}\exp(-\lambda) - \frac{2}{\lambda^2}\exp(-\lambda) + \frac{2}{\lambda^2} \right)$$

Then

$$I(\lambda) = \frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}.$$

Proposition 2.1. *For the $TESS(\lambda)$ models, the Fisher's information of the parameter λ satisfies the following properties:*

(i) $I(\lambda)$ is symmetric about 0.

(ii) For every $\lambda \neq 0, \quad I(\lambda) > 0$.

(iii) $I(\lambda)$ has a finite limit; in fact, $\lim_{\lambda\to 0} I(\lambda) = \frac{1}{12}$.

(iv) $I(\lambda) < \frac{1}{\lambda^2}$.

(v) $I(\lambda)$ does not depend on f and F.

Proof. See Appendix. □

Table 13: Comparison of Bayesian estimators for the *TESLD* versus *ASLD*, ($\sigma = 2$,$\lambda = 5$)

		SLD				*TESLD*			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	2.00424	0.01325	5.44160	1.29904	1.99497	0.00976	5.03592	0.18386
	500	1.99959	0.00569	5.15312	0.48395	1.99432	0.00368	5.06051	0.06507
	1000	2.00468	0.00264	5.06456	0.19461	1.99696	0.00174	5.02221	0.03843
LINEX,c=0.5	200	2.00072	0.01314	5.14957	0.81906	1.99250	0.00934	4.98704	0.17647
	500	1.99819	0.00568	5.04996	0.41242	1.99335	0.00369	5.04084	0.06221
	1000	2.00398	0.00262	5.01564	0.18103	1.99648	0.00175	5.01253	0.03783
LINEX,c=2	200	1.99035	0.01296	5.54059	0.63219	1.98521	0.00975	4.84621	0.18333
	500	1.99402	0.00567	4.74278	0.35170	1.99043	0.00371	4.98273	0.05834
	1000	2.00186	0.00260	4.48916	0.17100	1.99502	0.00175	4.98374	0.03717

Table 14: Comparison of Bayesian estimators for the *TESLD* versus *ASLD*, ($\sigma = 2$,$\lambda = 10$)

		SLD				*TESLD*			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	1.98941	0.01266	11.93032	17.62184	1.99302	0.00938	10.23105	1.04870
	500	2.01182	0.00593	10.46066	3.99724	2.00361	0.00317	10.7680	0.44267
	1000	1.99826	0.00220	10.42781	1.87022	1.99852	0.00160	10.05860	0.19022
LINEX,c=0.5	200	1.98954	0.01260	9.76911	4.56891	1.99083	0.00937	9.97973	0.91487
	500	2.01040	0.00588	9.75192	2.46564	2.00272	0.00316	9.97908	0.41398
	1000	1.99756	0.00220	10.06484	1.34317	1.99801	0.00160	10.01006	0.18202
LINEX,c=2	200	1.97572	0.01280	7.32993	8.30673	1.98435	0.00942	9.34152	1.07750
	500	2.00617	0.00576	8.40833	3.64098	2.00008	0.00314	9.70532	0.44178
	1000	1.99548	0.00220	9.21845	1.41002	1.99677	0.00160	9.86912	0.18576

According to the above facts, the Jeffreys' prior of λ is given by

$$\pi(\lambda) \propto \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}}. \tag{2.7}$$

Proposition 2.2. *The Jeffreys' prior $\pi(\lambda)$ satisfies the following properties:*

(i) $\pi(\lambda)$ is symmetric about $\lambda = 0$ and it is decreasing in $|\lambda|$.

(ii) The tails of $\pi(\lambda)$ are of order $O(|\lambda|^{-1})$.

(iii) $\pi(\lambda)$ is improper.

(iv) $I(\lambda)$ does not depend on f and F.

Proof. See Appendix. □

Figure 1 illustrates the tail behavior, symmetry, and improperness of the Jeffreys' prior of λ. Let $y_1, y_2, \ldots, y_n$ be a random sample with $Y \sim TESS(\lambda)$. The posterior distribution of λ using the Jeffreys' prior (2.7) is given by

$$\pi(\lambda|y) \propto \pi(\lambda) \times L(\lambda) = \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}} \left(\frac{\lambda}{1-\exp(-\lambda)}\right)^n \times \prod_{i=1}^{n} f_X(y_i) \exp\{-\lambda F_X(y_i)\}.$$

Although $\pi(\lambda)$ is improper, in the next section, we have shown that it yields a proper posterior distribution for some distributions. The proof of whether these priors lead to proper or improper posteriors depends on the choice of functions f and F in (1.5).

In continuation, we consider the class of skew distributions (1.5) to derive the Fisher information of μ, σ, and λ.

Proposition 2.3. *The Jeffreys' priors associated with the model (1.5) are*

$$\pi(\mu) \propto 1, \quad \pi(\sigma) \propto \frac{1}{\sigma},$$

$$\pi(\lambda) \propto \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}}$$

Proof. See Appendix. □

Accordingly, the independence Jeffreys' prior of $= (\mu, \sigma, \lambda)'$ corresponding to model (1.5) is given by

$$\pi_I() = \pi_I(\mu, \sigma, \lambda) = \frac{1}{\sigma}\pi(\lambda). \tag{2.8}$$

Proposition 2.4. *For the $TESS(\mu, \sigma, \lambda)$ model, the $\pi_I(\mu, \sigma, \lambda)$ satisfies the following properties:*

(i) $\pi_I(\mu, \sigma, \lambda)$ is symmetric about $\lambda = 0$.

(ii) The tails of $\pi_I(\mu, \sigma, \lambda)$ are of order $O(|\lambda|^{-1})$.

(iii) $\pi_I(\mu, \sigma, \lambda)$ is improper.

Proof. The proof is similar to the proof of the proposition(2.2). □

Let $y_1, y_2, \ldots, y_n$ be a random sample with $Y \sim TESS(\mu, \sigma, \lambda)$. Using the prior structure in (2.8), the corresponding joint posterior distribution for μ, σ, λ is

$$\begin{aligned} \pi(\mu, \sigma, \lambda|) &\propto L(\mu, \sigma, \lambda)\pi_I(\mu, \sigma, \lambda) \\ &= \frac{\lambda^n}{\sigma^{n+1}(1-\exp(-\lambda))^n} \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}} \\ &\times \prod_{i=1}^{n} f_X\left(\frac{y_i - \mu}{\sigma}\right) \exp\left(-\lambda F_X\left(\frac{y_i - \mu}{\sigma}\right)\right) \end{aligned} \tag{2.9}$$

By considering the Bayesian analysis under improper priors, it is important to check if these priors yield proper posterior distributions or not. For $TESS$ models, although the $\pi_I(\mu, \sigma, \lambda)$ is improper, we will show that it leads to posterior distribution for some densities.
To achieve the purpose, by using the logistic PDF and CDF, $f(t) = \frac{\exp(-t)}{(1+\exp(-t))^2}$ and $F(t) = \frac{1}{1+\exp(-t)}$ in (1.5), we construct the truncated-exponential skew-logistic (TESL) distributions.

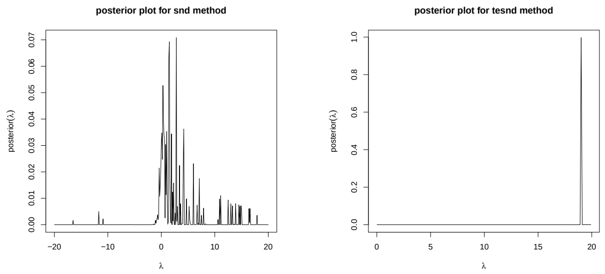

Figure 5: Posterior plot for $ASND$, $TESND$.

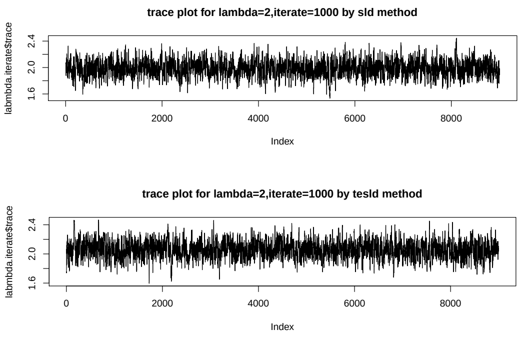

Figure 6: Trace plot for SLD and $TESLD$ with $\lambda = 2$

3 Bayesian inference for $TESL$ distribution

Mirzadeh and Iranmanesh [18] introduced a new skew-logistic distribution which is based on $TESS$ distribution, called $TESL$ distribution.

Definition 3.1. *A random variable Y has the TESL distribution with parameters $\mu \in \mathbb{R}, \sigma > 0$ and $\lambda \in \mathbb{R}$, denoted by $TESL(\mu, \sigma, \lambda)$, if its PDF is given by*

$$f_Y(y) = \frac{\lambda \exp[-(y-\mu)/\sigma]}{\sigma[1-\exp(-\lambda)][1+\exp[-(y-\mu)/\sigma]]^2} \times \exp\left(\frac{-\lambda}{1+\exp[-(y-\mu)/\sigma]}\right), y \in \mathbb{R}. \quad (3.10)$$

Then the CDF of Y is

$$F_Y(y) = \left[\frac{1}{1-\exp(-\lambda)}\right] \times \left[1-\exp\left(\frac{-\lambda}{1+\exp[-(y-\mu)/\sigma]}\right)\right], y \in \mathbb{R}.$$

If $\mu = 0$ and $\sigma = 1$, then it is denoted by $Y \sim TESL(\lambda)$.

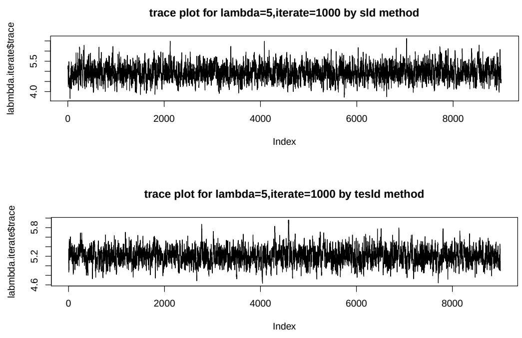

Figure 7: Trace plot for SLD and $TESLD$ with $\lambda = 5$

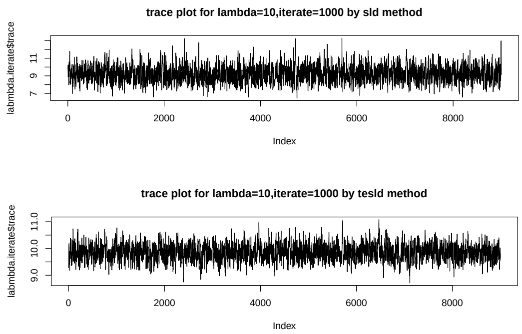

Figure 8: Trace plot for SLD and TESLD with $\lambda = 10$

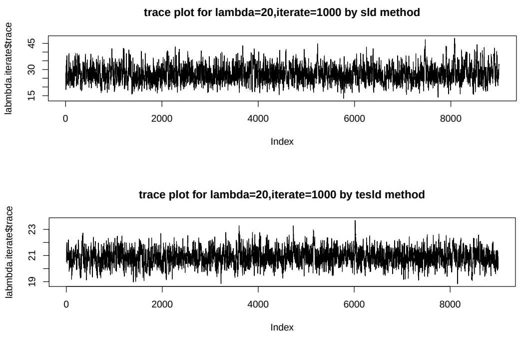

Figure 9: Trace plot for SLD and TESLD with $\lambda = 20$

Let $y_1, y_2, \ldots, y_n$ be a random sample with $Y \sim TESL(\lambda)$. The posterior distribution of λ using the (2.8) is given by

$$\begin{aligned}\pi(\lambda|y) &\propto \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}} \left(\frac{\lambda}{1-\exp(-\lambda)}\right)^n \\ &\times \prod_{i=1}^{n} \frac{\exp(-y_i)}{(1+\exp(-y_i))^2} \\ &\times \exp\left(-\lambda \sum_{i=1}^{n} \frac{1}{1+\exp(-y_i)}\right) \\ &\propto \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}} \left(\frac{\lambda}{1-\exp(-\lambda)}\right)^n \\ &\times \prod_{i=1}^{n} \exp\left(\sum_{i=1}^{n} \frac{-\lambda}{1+\exp(-y_i)}\right). \qquad (3.11)\end{aligned}$$

Figure 2 shows that the posterior distribution derived in (3.11) is proper. Empirically, we have found that $\pi(\lambda|\boldsymbol{y})$ can be reasonably well approximated by a $TESL(\lambda)$ distribution. Figure 3 illustrates the quality of this approximation.
Let $y_1, y_2, \cdots, y_n$ be a random sample with $Y \sim TESL(\mu, \sigma, \lambda)$. The corresponding joint the posterior distribution for μ, σ, λ associated to (3.10) with the independence Jeffreys' prior in (2.8) according to (2.9) is derived as follows:

$$\begin{aligned}\pi_I(\mu, \sigma, \lambda|) &\propto \frac{\lambda^n}{\sigma^{n+1}(1-\exp(-\lambda))^n} \\ &\times \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}} \\ &\times \exp\left(-\lambda \sum_{i=1}^{n} \frac{1}{1+\exp\left(-\left(\frac{y_i-\mu}{\sigma}\right)\right)}\right) \\ &\times \prod_{i=1}^{n} \left[\frac{\exp\left(-\left(\frac{y_i-\mu}{\sigma}\right)\right)}{\left(1+\exp\left(-\left(\frac{y_i-\mu}{\sigma}\right)\right)\right)^2}\right] \qquad (3.12)\end{aligned}$$

In the simulations, we describe later, we compare the Bayesian estimator of skewness parameters in (3.10) and Azzalini skew-logistic distribution ($ASLD$). For this purpose, we are

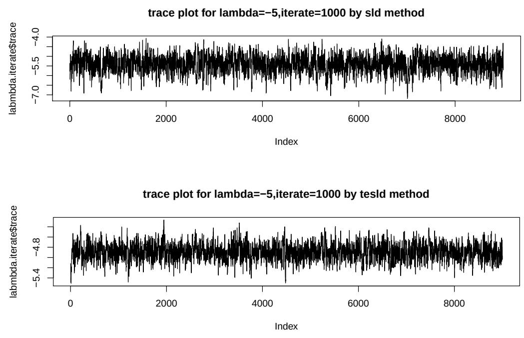

Figure 10: Trace plot for SLD and TESLD with $\lambda = -5$

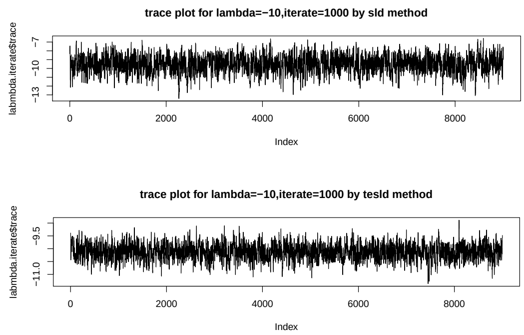

Figure 11: Trace plot for SLD and TESLD with $\lambda = -10$

Table 15: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\sigma = 5$,$\lambda = 5$)

		SLD				TESLD			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	5.04642	0.10843	5.53783	2.97721	4.96300	0.06931	5.11728	0.22703
	500	4.99845	0.03771	5.01659	0.42651	5.00000	0.02202	5.02283	0.06483
	1000	5.03370	0.01700	5.11000	0.20862	4.99049	0.01533	5.00565	0.03908
LINEX,c=0.5	200	5.02422	0.10502	5.20664	1.65934	4.94787	0.06971	5.06711	0.21019
	500	4.98968	0.03775	4.92082	0.38841	4.99386	0.02212	5.00343	0.06345
	1000	5.02926	0.16610	5.06002	0.19000	4.98743	0.01535	4.99602	0.03880
LINEX,c=2	200	4.96067	0.10089	4.56548	0.91452	4.90412	0.07350	4.92269	0.19182
	500	4.96390	0.03798	4.67132	0.39669	4.97573	0.02232	4.94613	0.06385
	1000	5.01609	0.01589	4.92030	0.16689	4.97833	0.01553	4.96737	0.03907

Table 16: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\sigma = 5$,$\lambda = 10$)

		SLD				TESLD			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	5.02249	0.79740	11.42915	17.09594	4.99753	0.05670	10.20417	1.04413
	500	4.97801	0.02402	10.27023	3.84931	4.99911	0.02374	9.99603	0.35719
	1000	4.99852	0.01289	10.30967	1.62729	4.99188	0.00918	10.09494	0.18183
LINEX,c=0.5	200	5.00047	0.07782	9.45900	5.09031	4.98376	0.05637	9.95391	0.87437
	500	4.96935	0.02431	9.58922	2.58630	4.99364	0.02368	9.90112	0.34863
	1000	4.99415	0.01288	9.95839	1.23363	4.98915	0.00922	10.04787	0.01703
LINEX,c=2	200	4.93742	0.07782	7.17721	9.20107	4.94380	0.05760	9.31910	1.08217
	500	4.94862	0.02602	8.28076	4.03900	4.97744	0.02385	9.63493	0.42581
	1000	4.98116	0.01307	9.13472	1.49384	4.98099	0.00940	9.90584	0.16460

Table 17: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, ($\sigma = 4$,$\lambda = 3$)

		SLD				TESLD			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	4.01081	0.04550	3.07077	0.43685	4.00820	0.04364	2.97680	0.11013
	500	4.00466	0.02340	3.02762	0.12599	3.99760	0.01678	3.01089	0.03437
	1000	3.99703	0.00935	3.02206	0.04597	4.00655	0.01002	3.01653	0.01720
LINEX,c=0.5	200	3.99682	0.04476	3.00053	0.37300	3.99663	0.04303	2.95205	0.01102
	500	3.99906	0.02325	3.00149	0.11994	3.99300	0.01674	3.00090	0.03407
	1000	3.99424	0.00934	3.00920	0.44629	4.00424	0.00997	3.01155	0.01701
LINEX,c=2	200	3.95644	0.04488	2.82242	0.29575	3.96307	0.04281	2.87905	0.11792
	500	3.98250	0.02317	2.92776	0.11141	3.97941	0.01687	2.97113	0.03436
	1000	3.98594	0.00943	2.97176	0.04271	3.97738	0.00988	2.99664	0.01674

Table 18: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\sigma = 10, \lambda = 8)$

		SLD				TESLD			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	10.00894	0.33622	8.98437	9.16560	9.98114	0.21323	8.16645	0.52855
	500	9.99037	0.13254	8.27578	1.93936	10.00469	0.07242	8.03113	0.23142
	1000	10.03578	0.06272	8.12262	0.69213	9.99467	0.05019	8.04506	0.10220
LINEX,c=0.5	200	9.92276	0.0.33040	7.85335	3.42616	9.92637	0.21353	8.02764	0.45853
	500	9.95566	0.13256	7.89862	1.37859	9.98260	0.07205	7.97761	0.22321
	1000	10.01825	0.06133	7.94455	0.59879	9.98363	0.05019	8.01809	0.09879
LINEX,c=2	200	9.68602	0.39283	6.26833	4.078583	9.77248	0.24710	7.65296	0.48014
	500	9.85519	0.14627	7.07989	1.56724	9.91866	0.07661	7.82362	0.23286
	1000	9.96666	0.06081	7.48892	0.68940	9.95099	0.05164	7.93882	0.09737

Table 19: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\sigma = 8, \lambda = 15)$

		SLD				TESLD			
	n	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	7.92576	0.22147	32.42068	17070.76	8.01771	0.14518	15.42787	3.36551
	500	8.00166	0.07987	16.49566	18.33011	7.99866	0.05140	15.10509	1.04971
	1000	7.97288	0.05580	15.53666	4.71674	7.98220	0.02729	15.16757	0.65452
LINEX,c=0.5	200	7.87122	0.22678	13.05701	13.14064	7.98164	0.14300	14.66550	2.48041
	500	7.97936	0.07936	14.12462	6.96667	7.98427	0.05133	14.81020	0.96155
	1000	7.96179	0.05621	14.44395	3.19855	7.97503	0.02753	15.01416	0.59143
LINEX,c=2	200	7.71908	0.27399	8.79392	40.10286	7.87880	0.15133	13.04517	5.14057
	500	7.91447	0.08361	11.02269	17.8028	7.94200	0.05355	14.03639	1.61696
	1000	7.92902	0.05887	12.37616	8.18808	7.95375	0.02876	14.58678	0.67353

Table 20: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\mu = 3, \sigma = 5, \lambda = 2)$

		SLD						TESLD					
	n	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	3.02427	0.10434	4.99331	0.09699	2.08798	0.14915	3.05689	0.36565	4.98488	0.08378	1.98544	0.05720
	500	3.01538	0.05770	5.00025	0.03435	2.05131	0.05007	2.97193	0.10074	4.98536	0.03106	2.01801	0.03065
	1000	3.05362	0.03273	5.00994	0.01505	2.00967	0.02238	2.97743	0.07631	4.99591	0.01681	2.00103	0.01129
LINEX,c=0.5	200	2.98297	0.10415	4.97154	0.09601	2.06014	0.13603	2.97347	0.36297	4.96486	0.08342	1.96641	0.05724
	500	2.99898	0.05746	4.99151	0.03418	2.04074	0.04802	2.93842	0.10380	4.97752	0.03115	2.01031	0.03023
	1000	3.04541	0.03191	5.00556	0.01493	2.00467	0.02206	2.96072	0.07733	4.99198	0.01680	1.99721	0.01128
LINEX,c=2	200	2.85778	0.12446	4.90930	0.09869	1.98372	0.11114	2.72352	0.43920	4.90848	0.08681	1.909820	0.06329
	500	2.94958	0.05997	4.96578	0.03458	2.01007	0.04349	2.83793	0.12641	4.95440	0.03213	1.98732	0.03008
	1000	3.02075	0.03026	4.99254	0.01480	1.98991	0.02143	2.91059	0.08371	4.98028	0.01695	1.98577	0.01141

Table 21: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\mu = 4, \sigma = 8, \lambda = 5)$

		SLD						TESLD					
	n	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	3.93132	0.21399	8.07505	0.27722	5.48714	1.66793	3.88231	0.56937	8.00813	0.13428	4.97845	0.20542
	500	4.02378	0.08766	7.98463	0.09879	5.21085	0.58214	4.02194	0.32489	7.98375	0.05737	5.05653	0.07490
	1000	4.02021	0.03693	8.00275	0.04742	5.01910	0.19519	3.97831	0.09088	7.97384	0.02752	5.01584	0.04464
LINEX,c=0.5	200	3.87874	0.22380	8.018531	0.26441	5.18751	1.05320	3.72524	0.63130	7.96890	0.13219	4.93046	0.20307
	500	4.00308	0.08711	7.96243	0.09887	5.10362	0.48841	3.95939	0.32623	7.96816	0.05761	5.03691	0.07212
	1000	4.00981	0.03662	7.99157	0.04721	4.97151	0.18579	3.94684	0.09326	7.97602	0.02772	5.00619	0.04414
LINEX,c=2	200	3.71744	0.28845	7.86101	0.26356	4.56727	0.70113	3.25836	1.10542	7.85778	0.14342	4.79203	0.22365
	500	3.94038	0.09068	7.89783	0.10480	4.82758	0.38228	3.77247	0.37699	7.92246	0.06114	4.97904	0.06847
	1000	3.97846	0.03699	7.95855	0.04806	4.83844	0.18620	3.85254	0.11223	7.95285	0.02903	4.97744	0.04377

Table 22: Comparison of Bayesian estimators for the $TESLD$ versus $ASLD$, $(\mu = 7, \sigma = 6, \lambda = 10)$

		SLD						TESLD					
	n	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$	μ_b	$MSE(\mu_b)$	σ_b	$MSE(\sigma_b)$	λ_b	$MSE(\lambda_b)$
SEL	200	6.97989	0.06609	6.04743	0.12157	12.87472	119.4780	7.05049	0.22990	5.95362	0.08800	10.28370	1.15329
	500	6.99802	0.02075	5.98149	0.04830	10.66084	4.37395	7.01218	0.11262	6.01728	0.03824	10.08414	0.29090
	1000	7.01061	0.01101	5.99628	0.02949	10.42005	1.79222	7.01602	0.05816	6.00212	0.01150	10.07182	0.16527
LINEX,c=0.5	200	6.96360	0.06699	6.01567	0.11711	9.80833	6.09643	6.98448	0.22740	5.93418	0.08914	10.02852	0.92780
	500	6.19979	0.02079	5.96898	0.04851	9.92210	2.55373	6.98567	0.11268	6.00933	0.03782	9.98671	0.26942
	1000	7.00740	0.01096	5.99002	0.02946	10.05800	1.30438	7.00277	0.05793	5.99816	0.01147	10.02293	0.15644
LINEX,c=2	200	6.91365	0.07309	5.92566	0.11576	7.33651	8.54873	6.78867	0.27119	5.87809	0.09675	9.37942	1.02469
	500	6.97239	0.02141	5.93228	0.05094	8.50802	3.34942	6.90663	0.12120	5.98587	0.03734	9.71262	0.31437
	1000	6.99774	0.01091	5.97145	0.02983	9.21066	1.41380	6.96313	0.05934	5.98639	0.01156	9.88123	0.15847

reminded of Azzalini skew-logistic distribution, termed $X \sim SL(\mu, \sigma, \lambda)$ as a special case of (1.1) with PDF

$$f_X(x) = \frac{2}{\sigma} \frac{\exp(-\frac{x-\mu}{\sigma})}{(1+\exp(-\frac{x-\mu}{\sigma}))^2} \frac{1}{1+\exp(-\lambda \frac{x-\mu}{\sigma})}. \tag{3.13}$$

For $(\mu, \sigma) = (0, 1)$, it is denoted by $SL(\lambda)$. The Jeffreys' prior and the posterior distribution associated with $SL(\lambda)$ are given by

$$\pi(\lambda) \propto \sqrt{\int_0^\infty x^2 Sech^2\left(\frac{x}{2}\right) Sech^2\left(\frac{\lambda x}{2}\right) dx}, \tag{3.14}$$

$$\begin{aligned} \pi(\lambda|y) \quad &\propto \quad \sqrt{\int_0^\infty x^2 Sech^2\left(\frac{x}{2}\right) Sech^2\left(\frac{\lambda x}{2}\right) dx} \\ &\times \quad \prod_{i=1}^n \frac{1}{1+\exp(-\lambda y_i)}. \end{aligned}$$

Rubio and Liseo [22] proved that the $\pi(\lambda)$ is proper, so it follows that the $\pi(\lambda|y)$ is proper, too.
Similarly, the independence Jeffreys' prior of (μ, σ, λ) corresponding to model (3.13) is given by

$$\begin{aligned} \pi_I(\mu, \sigma, \lambda) \quad &\propto \quad \frac{1}{\sigma} \pi(\lambda) \\ &= \quad \frac{1}{\sigma} \sqrt{\int_0^\infty x^2 Sech^2\left(\frac{x}{2}\right) Sech^2\left(\frac{\lambda x}{2}\right) dx} \end{aligned} \tag{3.15}$$

and the posterior distribution is

$$\begin{aligned} \pi(\mu, \sigma, \lambda|y) \quad &\propto \\ &\frac{1}{\sigma^{n+1}} \sqrt{\int_0^\infty x^2 Sech^2\left(\frac{x}{2}\right) Sech^2\left(\frac{\lambda x}{2}\right) dx} \\ \times \quad &\prod_{i=1}^n \frac{\exp\left(-\frac{y_i-\mu}{\sigma}\right)}{(1+\exp\left(-\frac{y_i-\mu}{\sigma}\right))^2} \frac{1}{1+\exp\left(-\lambda \frac{y_i-\mu}{\sigma}\right)}. \end{aligned}$$

4 Simulation analysis

To illustrate the results of the previous sections, we now apply our findings in three examples. In examples (4) and (4) we computed Bayesian estimators for the skewness parameter λ and in example (4), we computed Bayesian estimators for the location parameter μ, scale parameter σ and skewness parameter λ when the error loss function is square error loss (SEL) and Linex loss function (for $c = 0.5, 2$) is defined by

$$L(\theta, \widehat{\theta}) = (\theta - \widehat{\theta})^2, \widehat{\theta} \in D, \ \theta \in \Theta.$$

$$\begin{aligned} L(\theta, \widehat{\theta}) = b\left[e^{a(\widehat{\theta}-\theta)} - a(\widehat{\theta}-\theta) - 1\right], \\ a \neq 0, \ b > 0, \ \widehat{\theta} \in D, \ \theta \in \Theta. \end{aligned}$$

For this purpose, we generated samples of different sizes from the desired distribution. We repeated this process one hundred times and computed the average of the estimates, the average of the bias (Ab), and the average of the mean squared error (MSE). As we know that under SEL the Bayesian estimators for λ and σ are given by

$$\begin{aligned} \lambda_b &= E(\lambda|) = \int_{R-\{0\}} \lambda\pi(\lambda|)d\lambda, \\ \sigma_b &= E(\sigma|) = \int_{R-\{0\}} \sigma\pi(\sigma|)d\sigma, \end{aligned} \tag{4.16}$$

and that under the Linex loss function, the Bayesian estimators for λ and σ are given by

$$\begin{aligned} \lambda_b &= \frac{-1}{c}\log E(\exp(-c\lambda)|) \\ &= \frac{-1}{c}\log\left(\int_{R-\{0\}} \exp(-c\lambda)\pi(\lambda|)\,d\lambda\right), \\ \sigma_b &= \frac{-1}{c}\log E(\exp(-c\sigma)|) \\ &= \frac{-1}{c}\log\left(\int_{R-\{0\}} \exp(-c\sigma)\pi(\sigma|)\,d\sigma\right). \end{aligned} \tag{4.17}$$

As expected, the posterior densities have complicated integration. Therefore, the Markov chain Monte Carlo (MCMC) method was considered to obtain the posterior estimates. Gaining the goal, we used the Metropolis–Hastings algorithm to simulate the posterior quantities. For each simulated data set, 50000 iterations were performed using MCMC methods. As a burn-in, the first 1000 initial values were discarded. As a first example, we consider the above method for $TESL(\lambda)$ and $SL(\lambda)$, that is, we assume that $\mu = 0$ and $\sigma = 1$. For this purpose, we generated samples of size $n = 100, 200, 500, 1000$ for $\lambda = -5, -10, 2, 5, 10, 20$ from $TESL(\lambda)$ and $SL(\lambda)$.
The results are represented in Tables $1-6$. According to those tables, the biases and the mean squared errors in $TESL(\lambda)$ approach zero with increasing n, but the result in $ASL(\lambda)$ is not good. Especially for large values of λ, the Bayesian estimator of λ is much better in $TESLD$. Figure 1 shows the generated posterior density in $TESLD$ and $ASLD$ models by MCMC methods. That Figure suggests a better convergence of the $TESL$ model. Figures $6-11$ show the trace plot of $TESLD$ and $ASLD$ for different values of λ. These Figures suggest a better convergence of the $TESL$ model. As a second example, we calculate the Bayesian estimators for parameters in the truncated-exponential skew-normal distribution ($TESND$) and Azzalini skew-normal distribution ($ASND$). We simulated samples of size $n = 100, 200, 500, 1000$ from $TESND$ and $ASND$ with parameters $\mu = 0$, $\sigma = 1$ and $\lambda = -10, -5, 2, 5, 10, 20$. The results are reported in Tables $7-12$. Overall, the biases and the mean squared errors of estimators in $TESND$ are better than those in $ASND$, especially for large values of λ. Figure 5 shows the generated posterior density in $TESND$ and $ASND$ models by MCMC methods. That Figure suggests a better convergence of the $TESN$ model. In this example, we compare the performances of the Bayesian estimator of (μ, σ, λ) when X is $TESL(\mu, \sigma, \lambda)$ and $SL(\mu, \sigma, \lambda)$ distribution, presented in Section 3. For this purpose, we generated samples of size $n = 200, 500, 1000$ from (3.10) and (3.13) for $(\mu, \sigma, \lambda) = (0, 2, 5), (0, 2, 10), (0, 5, 5), (0, 5, 10), (0, 4, 3)$ $, (0, 10, 8), (0, 8, 15), (3, 5, 2), (4, 8, 5), (7, 6, 10)$.
To compute estimators, $\pi_I(\sigma, \lambda|)$ is replaced in (4.16) and (4.17). The results are reported in Tables $13-22$. Those tables show how the mean squared errors for the $TESLD$ are smaller than SLD. Also as shown in these tables, the Bayesian estimator in $TESLD(\lambda)$ becomes more accurate than the $SLD(\lambda)$ model as the absolute value of skewness parameter increases. In general, the use of a $TESSD$ seems appropriate to model asymmetric data. This is because of the evidence presented in Tables $1-22$ that estimators based on $TESS$ distributions have smaller biases and smaller mean squared errors than estimators based on Azzalini skew distributions. R codes for these examples are available upon request.

5 Conclusion

We have studied the Jeffreys' prior of the skewness parameter of $TESS$ models as well as the independence Jeffreys' prior for the same class models with unknown location and scale parameters. We have also investigated the properties of this proper distribution such as symmetry, improperness, and the order of tails. We have also presented the existence of the posterior distribution for some subclass of $TESS$ models such as the truncated-exponential skew-normal and the truncated exponential skew-logistic distributions. In the simulation studies, it was proved that the Bayesian estimators for parameters based on the Jeffreys' prior in $TESS$ distributions can be expected to be better than the Azzalini skew distributions.

Appendix

Proof of Proposition 2.1.
(i) Since $I(\lambda) = I(-\lambda)$, $I(\lambda)$ is symmetric about 0.
(ii)

$$\begin{aligned} I(\lambda) &= \frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2} \\ &= \frac{[1-\exp(-\lambda)]^2 - \lambda^2 \exp(-\lambda)}{\lambda^2(1-\exp(-\lambda))^2} \end{aligned}$$

Since $[1-\exp(-\lambda)]^2 > \lambda^2 \exp(-\lambda)$ the proof is complete.
(iii)We have

$$\lim_{\lambda\to 0} I(\lambda) = \lim_{\lambda\to 0}\left(\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}\right),$$

by applying the $L'Hopital's$ rule four times, we get $\lim_{\lambda\to 0} I(\lambda) = \frac{1}{12}$.
(iv) Note that since $\frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2} > 0$, then

$$0 < I(\lambda) = \frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2} < \frac{1}{\lambda^2}. \quad (5.18)$$

Therefore $I(\lambda)$ is upper bounded. It follows that $I(\lambda)$ has tails of order $O(|\lambda|^{-2})$.
(v) We see that $I(\lambda)$ does not depend on f and F.

Proof of Proposition 2.2.
(i) $\pi(\lambda) = \pi(-\lambda)$ then $\pi(\lambda)$ is symmetric about $\lambda = 0$. Since $\frac{d\pi(\lambda)}{d\lambda} < 0$, then $\pi(\lambda)$ is decreasing in $|\lambda|$.
(ii) According to (5.18), $0 < \pi(\lambda) < \frac{1}{\lambda}$. Then the tails of $\pi(\lambda)$ are of order $O(|\lambda|^{-1})$.
(iii) Note that since $\pi(\lambda)$ is undefined at $\lambda = 0$ and the tails of $\pi(\lambda)$ are of order $O(|\lambda|^{-1})$, $\pi(\lambda)$ is improper.
(iv) We see that $I(\lambda)$ does not depend on f and F.

Proof of Proposition 2.3.
The logarithm of (1.5) is given by

$$\begin{aligned} \log(f_Y(y;\mu,\sigma,\lambda)) &= \log\left(\frac{\lambda}{1-\exp(-\lambda)}\right) - \log(\sigma) \\ &+ \log(f_X(t)) - \lambda F_X(t), \end{aligned}$$

where $t = \frac{y-\mu}{\sigma}$. The Fisher information of μ, σ, and λ can be written as follows:

$$\begin{aligned} I(\mu) &= E\left(\frac{\partial}{\partial\mu}\log(f_Y(Y;\mu,\sigma,\lambda))\right)^2 \\ &= \frac{1}{\sigma^2}E\left(\frac{f'_X(T)}{f_X(T)} + \lambda f_X(T)\right)^2 \\ &= \frac{\lambda}{\sigma^2(1-\exp(-\lambda))} \\ &\times \int_{-\infty}^{\infty}\left(\frac{f'_X(t)}{f_X(t)} + \lambda f_X(t)\right)^2 f_X(t) \\ &\times \exp\left(-\lambda F_X(t)\right) dx, \end{aligned}$$

$$\begin{aligned} I(\sigma) &= E\left(\frac{\partial}{\partial\sigma}\log(f_Y(Y;\mu,\sigma,\lambda))\right)^2 \\ &= \frac{1}{\sigma^2}E\left(1 + T\frac{f'_X(T)}{f_X(T)} - \lambda T f_X(T)\right)^2 \\ &= \frac{\lambda}{\sigma^2(1-\exp(-\lambda))} \\ &\times \int_{-\infty}^{\infty}\left(1 + t\frac{f'_X(t)}{f_X(t)} - \lambda t f_X(t)\right)^2 f_X(t) \\ &\times \exp\left(-\lambda F_X(t)\right) dx, \end{aligned}$$

$$\begin{aligned} I(\lambda) &= E\left(\frac{\partial}{\partial\lambda}\log(f_Y(Y;\mu,\sigma,\lambda))\right)^2 \\ &= E\left(\frac{1}{\lambda} - \frac{\exp(-\lambda)}{1-\exp(-\lambda)} - F_X(T)\right)^2 \\ &= E\left(\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}\right) \\ &= \frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}. \end{aligned}$$

Thus the Jeffreys' priors associated with this model are

$$\pi(\mu) \propto 1, \quad \pi(\sigma) \propto \frac{1}{\sigma}, \quad \pi(\lambda) \propto \sqrt{\frac{1}{\lambda^2} - \frac{\exp(-\lambda)}{(1-\exp(-\lambda))^2}}.$$

References

[1] A. Azzalini, A class of distributions include the normal ones, *Scandinavian Journal of Statistics* 12 (1985) 171-178.

[2] A. Azzalini, Further results on a class of distributions which includes the normal ones, *Statistica* XLVI (1986) 199-208.

[3] A. Azzalini, A. Capitanio, Statistical applications of the multivariate skew normal distribution, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 61 (1999) 579-602.

[4] A. Azzalini, A. Capitanio, Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 65 (2003) 367-389. doi.org/10.1111/1467-9868.00391

[5] A. Azzalini, M. G. Genton, Robust likelihood methods based on the skew-t and related distributions, *International Statistical Review* 76 (2008) 106-129, doi.org/10.1111/j.1751-5823.2007.00016.x

[6] M. D. E. Branco, M. G. Genton, B. Liseo, Objective Bayesian analysis of skew-t distributions, *Scandinavian Journal of Statistics* 40 (2013) 63-85, doi:101111/j.1467.9469.2011.00779.x

[7] S. Cabras, W. Racugno, M. E. Castellanos, L. Ventura, A matching prior for the shape parameter of the skew-normal distribution, *Scandinavian Journal of Statistics* 39 (2012) 236-247, doi.org/10.1111/j.1467-9469.2011.0775.x

[8] A. Canale, E. C. Kenne Pagui, B. Scarpa, Bayesian modeling of university first-year students' grades after placement test, *Journal of Applied Statistics* 43 (2016) 3015-3029, doi.org/10.1080/02664763.2016.1157144

[9] H. Dette, C. Ley, F. Rubio, Natural (Non) Informative Priors for Skew-symmetric Distributions, *Scandinavian Journal of Statistics* 45 (2018) 405-420. doi.org/10.1111/sjos.12306

[10] J. T. A. S. Ferreira, M. F. J. Steel, A constructive representation of univariate skewed distributions, *Journal of the American Statistical Association* 101 (2006) 823-829. doi.org/10.1198/016214505000001212

[11] F. Ghaderinezhad, C. Ley and N. Loperfido, Bayesian Inference for Skew-Symmetric Distributions, *Symmetry* 12 (2020) 491-505, doi.org/10.3390/sym12040491

[12] M. Hallin, C. Ley, Skew-symmetric distributions and Fisher information −a tale of two densities, *Bernoulli* 18 (2012) 747-763, doi.org/10.3150/12-BEJ346

[13] M. C. Jones, On families of distributions with shape parameters (with discussion), *International Statistical Review* 83 (2015) 175-192, doi.org/10.1111/insr.12055

[14] C. Ley, Flexible modelling in statistics: past, present and future, *Journal de la Socit Franaise de Statistique* 156 (2015) 76-96.

[15] C. Ley, D. Paindaveine, On the singularity of multivariate skew-symmetric models, *Journal of Multivariate Analysis* 101 (2010) 1434-1444, doi.org/10.1016/j.jmva.2009.10.008

[16] B. Liseo, N. Loperfido, Default Bayesian analysis of the skew-normal distribution, *Journal of Statistical Planning and Inference* 136 (2004) 373-389.

[17] B. Liseo, N. Loperfido, A note on reference priors for the scalar skew-normal distribution, *Journal of Statistical Planning and Inference* 136 (2006) 373-389. doi.org/10.1016/j.jspi.2004.06.062

[18] S. Mirzadeh, A. Iranmanesh, A new class of skew-logistic distribution, *Mathematical Sciences* 13 (2019) 375-385, doi.org/10.1007/s40096-019-00306-8

[19] S. Nadarajah, The skew-logistic distribution, *AStA Advances in Statistical Analysis* 93 (2009) 187-203, doi.org/10.1007/s10182-009-0105-6

[20] S. Nadarajah, V. Nassiri, A. Mohammadpour, Truncated-exponential skew-symmetric distributions, *Statistics* 48 (2014) 872-895, doi.org/10.1080/02331888.2013.821474

[21] A. Pewsey, Problems of inference for Azzalini's skewnormal distribution, *Journal of applied statistics* 27 (2000) 859-870, doi.org/10.1080/02664760050120542

[22] F. J. Rubio, B. Liseo, On the independence Jeffreys' prior for skew-symmetric models, *Statistics and Probability Letters* 85 (2014) 91-97, doi.org/10.1016/j.spl.2013.11.012

[23] J. Wang, J. Boyer, M. G. Genton, A skew-symmetric representation of multivariate distributions, *Statistica Sinica* 14 (2004) 1259-1270.

Saeed Mirzadeh is an instructor at the Islamic Azad University, Torbat Heydariyeh Branch. He received his M.S. in Applied Mathematics from K. N. Toosi University of Technology in 2001 and Ph.D. in statistics from Islamic Azad University, Mashhad Branch in 2021. His research interests are Distribution theory, Bayesian inference, Computational statistics, and Linear models

Anis Iranmanesh is an assistant professor at the Islamic Azad University, Mashhad Branch. She received her B.S., M.S., and Ph.D. in statistics from the Ferdowsi University of Mashhad in 1988, 1992, and 2012, respectively. Her research interests are Distribution theory, Multivariate analysis, Bayesian inference, and Nonparametric methods.

Ehsan Ormoz is an assistant professor at the Islamic Azad University, Mashhad Branch. He received his M.S. and Ph.D. in statistics from Allameh Tabatabai University and Islamic Azad University Science and Research Branch in 2005 and 2009, respectively. His research interests are Computational Statistics, Statistical learning, Variable selection, and Categorical data modeling.