



Developing the Methods of the Performance Evaluation of VaR Models Using Fuzzy TOPSIS and Copula-ARIMA-GARCH Model

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ABSTRACT

This study used the Copula-ARIMA-GARCH approach to calculate the VAR of a portfolio of four investment companies and daily data (from March/April 2009 to February/March 2017) to calculate risk more accurately and develop methods of the performance evaluation of value at risk (VaR¹) models by a combination of a copula function and ARIMA/GARCH models. In this study, a novel method was proposed to evaluate the performance of VaR models using the fuzzy multicriteria decision-making models. For this purpose, ranks attained by the VaR estimation models were employed considering the unconditional coverage test procedure, Dowd's loss function procedure, and the rank given the prediction accuracy. The results indicated that risk-taking and risk-indifferent investors assume that the VaR calculated from the Copula-ARIMA-GARCH model is the most accurate model, while risk-averse investors take the generalized extreme value (GEV²) model as the most accurate model considering the high importance of the loss function.

Keywords:

Value at Risk, Copula-ARIMA-GARCH, Generalized Extreme Value, Backtesting, Fuzzy TOPSIS

¹ Value at Risk

² Generalized Extreme Value

1. Introduction

Financial crises that have arisen in recent years indicated the inefficiency of traditional measurement and risk management techniques in volatile international markets in the 21st century. Given large fluctuations in the gold price, exchange rate, etc., over the last few years, Iranian companies must reconsider their risk management policies. For this purpose, they can use modern risk management tools such as copula functions as well as generally accepted criteria such as value at risk (VaR). Various tests have been presented for measuring the accuracy of VaR models since the late 1990s. Despite their difference in details, the majority of these tests have been founded upon the concept of comparing the actual loss with the reported VaR.

In this study, a novel model, i.e., the Copula-ARIMA-GARCH model, is proposed to measure market risk. Here, the ARIMA and GARCH models are respectively utilized to simultaneously explain and evaluate the mean and conditional variance. However, similar research is simply dedicated to explaining the conditional variance. Finally, the copula model is employed to simulate and calculate asset correlations in the portfolio more precisely. In the majority of the research published in prominent national and international journals, the sample portfolio includes two assets. In this research, however, our portfolio contains four assets. Furthermore, in most cases, only parameters are estimated but not predicted. Herein, we have used observation horizon prediction using estimated parameters, rendering this research more applied.

After calculating VaR using different traditional techniques, the Copula-GARCH model, the extreme value theory, they are compared in terms of estimation power. This study introduces a novel method for evaluating the performance of VaR models using fuzzy multicriteria decision-making models, which has not been dealt with in similar studies so far.

2. Literature Review

Investors take into consideration the project risk and return at the same time as one of the primary factors contributing to investment decisions when they are about to invest in different projects. It should be acknowledged that risk is an integral part of the return,

which must be taken into account when deciding the return of different investment projects.

Financial institutions mainly face the following risks: credit risk, liquidity risk, interest rate risk, market risk, off-balance-sheet risk, currency risk, political risk, and default risk. Risk managers primarily seek to strike a balance between risk and return.

Copula functions provide powerful tools to describe the structure of correlation between multidimensional random variables. For example, they are used to calculate the VaR of the portfolio in the field of risk management. Heteroscedasticity models are now widely used in time-series analysis, especially financial applications when subjected to analysis of variance (ANOVA) and estimation of variance components. Engle (1982) was the first to demonstrate that although most financial time series are unpredictable, significant clustering can be observed in their volatility. This phenomenon is called conditional heteroscedasticity because it is assumed that the series is generally stationary, but its expected conditional variance changes over time.

3. Research Background

Till Guldemann, head of research at J.P. Morgan's Corporate & Investment Bank (CIB), is believed to be the inventor of the term "value at risk (VaR)." In 1995, the Basel Committee on Banking Supervision (BCBS) obliged banks to use this model to determine their capital adequacy ratio (CAR).

In his research, Mills discovered for the first time that most economic variables lack normal distribution. High kurtosis and skewness in distribution indicate univariate non-normal distribution. Recent studies have also shown that multivariate distributions are non-normal, known as asymmetric dependence. An example of asymmetric dependence is when two assets have a higher correlation coefficient in a bear market than a bull market, illustrated in the works of Erb et al. (1994), Longin and Solnik (2001), and Ang and Chen (2002). This indicates the superiority of copula functions in modeling the correlation structure. Sklar (1959) was the first person who proposed to use copula functions to measure the nonlinear correlation between variables. Embrechts et al. (1999) were the first people who proposed to utilize copula in the financial area. Even though "copula" is a relatively new concept, advances in time-series modeling mainly

owe to higher-order conditional moments, such as Engle's (1982) Autoregressive Conditional Heteroskedasticity (ARCH) and Bollersleve's (1986) Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. Using these models, Patton (2001) introduced a conditional copula.

In their article entitled "Evaluating the Family of GARCH Models in Predicting Stock Market Fluctuations (case study: Tehran Stock Exchange)," Saeed Rasekhi et al. (2014) wrote that "the research findings suggested that first, t-distribution and general mistake outperformed normal distribution as they took excess kurtosis of return series into consideration, based on root-mean-square of prediction error (RMES) and Thiel's inequality coefficient (TIC). Second, PGARCH and GARCH models, assuming t-distribution, yielded the best performance, and EGARCH and TGARCH models, assuming a normal distribution, yielded the worst performance.

In a paper, Omari et al. (2018) calculated the VaR of the currency portfolio for four currency pairs using the GARCH-EVT-COPULA method. They concluded that the Student-t copula is the best copula in modeling the structure of correlation between all pairs of exchange rates. Moreover, the GARCH-EVT-COPULA model successfully implemented the VaR of the portfolio based on backtesting.

In a paper, Sahamkhadem et al. (2018) conducted portfolio optimization using the GARCH-EVT-COPULA forecasting models for ten stock indices. They concluded that the GARCH-EVT-COPULA forecasting models employ Gaussian and student-t copulas and have the best performance in risk mitigation.

In their paper, Huang and Chyan So (2018) proposed to apply the COPULA-GARCH model to estimate the VaR of a portfolio with a credit default swap (CDS). They selected six constant and time-varying copulas combined with GARCH skewed Student-t innovation (GARCH-skt) to form eight copula-GARCH models to capture the joint distribution of the two assets in the portfolio. They then computed corresponding 1-day VaRs. According to their findings, the time-varying symmetrized Joe-Clayton (SJC) copula model combined with the GARCH-skt (tvSJC-copula-GARCH-skt) performed best, regardless of the market situation.

In their paper, Wenhua Yu et al. (2017) examined the VaR and the expected shortfall in a given portfolio

composed of four crude oil assets using various GARCH models, extreme value theory, and vine copulas. The results demonstrated that combining the GARCH-type-EVT models and vine copulas can yield accurate risk criteria from the crude oil portfolio.

In a study entitled "examining the application of historical simulation methods in forecasting the expected shortfall using semi-parametric and non-parametric historical simulation models and parametric GARCH, EGARCH, and AGARCH models in four different asset groups using different sliding windows," Olsen (2015) evaluated the different risk prediction models. The results indicated that the semi-parametric historical simulation models and parametric GARCH-family models have a higher prediction capability than non-parametric historical simulation models.

In a paper, Tang et al. (2014) estimated the risk of the natural gas portfolio by mean of the GARCH-EVT-Copula model. They concluded that, for an equally weighted portfolio of five natural gases, the VaR and ES values obtained from the student-t copula are larger than those obtained from the Gaussian copula.

In addition, Ngoga Kirabo Bob (2013) used a combination of copula functions, extreme value theory, and GARCH to estimate the VaR of a portfolio composed of stock indices of Germany, Spain, Italy, and France. In their research, they demonstrated that the student-t copula had outperformed the other methods. Huang et al. (2009) calculated the VaR of a portfolio of the NASDAQ (National Association of Securities Dealers Automated Quotations system) and TAIEX (Technical Assistance and Information Exchange) indices using the copula-GARCH method. Palaro and Hotta (2006) calculated the VaR of a portfolio composed of the S&P 500 and NASDAQ indices using a mixed model with the conditional copula and multivariate GARCH to estimate the VaR of a portfolio composed of NASDAQ and S&P 500 indices. In this study, the VaRs of different methods are compared, indicating that the VaR obtained by the SJC copula has had the best performance.

In an article entitled "Forecasting the Value at Risk Using the GARCH-EVT Approach in Tehran Stock Exchange," Ali Dehghani et al. (2014) demonstrated that the approach to calculating VaR using the GARCH-EVT approach is more cost-effective and realistic. Similar research has failed to simultaneously

explain and estimate the mean and conditional variance by means of the ARIMA-GARCH model. They have also failed to investigate and compare the performance of the extreme value theory and the copula-GARCH model in calculating the VaR. On the other hand, this research offered a novel approach to evaluate the performance of VaR models using fuzzy multicriteria decision-making methods.

4. Methodology

This is an applied research study in purpose. Herein, by building a portfolio composed of the shares of four investment companies, with stock symbols A, B, C, and D, we seek to answer the following question, “Does the Copula-ARIMA-GARCH model perform better than other methods in comparing the portfolio risk or not?” For instance, this study used the data during an eight-year period (daily intervals from March/April 2009 to February/March 2017). After data reduction in Excel, a total of 1723 return-daily observations were eventually obtained for each share.

4.1. Marginal distribution model

The marginal distribution models considered in this study are based on ARIMA and GARCH models. Herein, to achieve i.i.d. distributions and estimate the variance, the [composite] mean and conditional variance model is explained and estimated simultaneously. To this aim, the Box-Jenkins methodology (ARIMA) is employed to model the mean of the return on assets and heteroscedasticity models (GARCH) to model the conditional variance. Their standardized error terms have a normal or Student’s t-distribution.

Using GARCH_n and GARCH_t models to estimate the variance

The normal GARCH model, an extension of Engle’s ARCH model, was introduced by Bollerslev. It is also known as the symmetric normal GARCH model. The symmetric normal GARCH model can be obtained utilizing unconditional variance as follows:

$$\begin{aligned} E(a_{t-1}^2) &= \sigma_{t-1}^2 \\ \sigma_t^2 &= \sigma_{t-1}^2 = \bar{\sigma}^2 \\ \bar{\sigma}^2 &= \frac{\alpha_2}{1-(\alpha_1+\alpha_2)} \quad (1) \end{aligned}$$

where α_1 represents the response of conditional variance to market shocks. When α_1 is large, conditional volatility highly responds to market shocks. To obtain the GARCH model parameters, the value of the following equation can be minimized:

$$-2 \ln L(\theta) = \sum_{t=1}^T \left(\ln(\sigma_t^2) + \left(\frac{a_t}{\sigma_t}\right)^2 \right) \quad (2)$$

where θ represents the parameters of the conditional variance equation for the symmetric normal GARCH model $\theta = (\omega, \alpha_1, \alpha_2)$.

Using the ARIMA(p, d, q) – GARCHn model to simultaneously estimate the mean and conditional variance

$$\begin{aligned} r_t &= \mu_t + \beta_1 r_{t-1} + \dots + \beta_p r_{t-p} + a_t + \gamma_1 a_{t-1} \\ &\quad + \dots + \gamma_q a_{t-q} \end{aligned}$$

$$\begin{aligned} a_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha_1 a_{t-1}^2 + \alpha_2 \sigma_{t-1}^2 \\ a_t | \Omega_{t-1} &\sim N(0, \sigma_t^2) \text{ or } t_d \end{aligned}$$

(3)

In this study, the maximum likelihood [estimation] method is employed to estimate the ARIMA-GARCH model parameters.

4.2. Copula functions

Let’s assume that X is a continuous random variable with domain D.

The [probability] density function X is the derivative of the distribution function $F(f(x) = F'(x))$. Since F is a monotonically increasing function, then $f(x) \geq 0$. The quantile of the continuous random variable X with the probability $\alpha \in [0,1]$ is the value of X_α from the random variable X, such that:

$$P(X < x_\alpha) = \alpha \quad (4)$$

Quantiles are used for simulation. In other words, to simulate a value for the random variable, a uniformly distributed random number is initially generated. Then, simulation is carried out using the inverse of the distribution function. To conduct simulation for

several random variables, copula functions capable of working with two variables must be used since our portfolio in this study consists of four assets (more than two assets). Thus, Gaussian copula functions are employed in this study, whose equation is as follows:

$$C_{\text{Gaussian}}(U_1, U_2; Q) = \Phi_Q(\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \quad (5)$$

To estimate VaR using copula functions, the return should be simulated by means of copula functions. Therefore, to estimate copula function parameters, marginal distribution parameters are initially estimated, followed by copula function parameters. However, the reverse is true in simulation using copula functions. That is, marginal distribution functions are simulated after correlations.

4.3. Simulation using conditional copulas

Let's assume that the joint distribution of the random variable X_1, \dots, X_n is obtained using n marginal distribution functions $F_i(x_i)$ and copula $C(u_1, \dots, u_n)$ ($x_i = F_i^{-1}(u_i)$). It is simulated using the following equation and joint distribution of variables:

- 1) The random number sequence $\{u_1, \dots, u_n\}$ is simulated using independent uniform random variables.
- 2) First, we set $u_1^* = u_1$. Then, using the inverse of the conditional copula $C_{2|1}^{-1}$, u_2 is converted to u_2^* as follows:

$$u_2^* = C_{2|1}^{-1}(u_2|u_1^*) \quad (6)$$

Now, using u_1^* and u_2^* , u_3^* can be calculated as follows:

$$u_3^* = C_{3|12}^{-1}(u_3|u_1^*, u_2^*) \quad (7)$$

By carrying out this operation for all u_i s, the set $\{u_1^*, \dots, u_n^*\}$ is obtained.

- 3) Using the inverse of the marginal distribution functions F_i^{-1} and the set $\{u_1^*, \dots, u_n^*\}$, the simulated values for each marginal distribution is obtained as $\{F_1^{-1}(u_1^*), \dots, F_n^{-1}(u_n^*)\}$.

4.4. Value at Risk

The VaR of a given portfolio at time t ($[t, \Delta t]$) at a $(1 - \alpha)$ confidence level where $\alpha \in (1,0)$ can be defined as follows:

$$P(X_{p,t} \leq VAR_t(\alpha) | \Omega_{t-1}) = \alpha \quad (8)$$

That is to say, we are $100(1 - \alpha)\%$ confident that the loss of our portfolio in the time interval between t and Δt will not exceed the VaR. Herein, our portfolio is composed of the shares of four investment companies. In this portfolio, the assets are equally weighted; however, the results obtained do not change with changing asset weights.

4.5. Estimating Value at Risk using the hybrid ARIMA(p, d, q) – GARCH_n model with the copula function

To estimate VaR using the copula-ARIMA-GARCH model, a model needs to be initially estimated for each marginal distribution. Mean-conditional variance models are applied to estimate the marginal distribution of investment companies introduced in this research. First, a model is estimated for each of the marginal distributions using the ARIMA-GARCH models. Afterward, the error terms of the models are standardized. If the standardized error term time-series of the marginal distributions is demonstrated as $\{\eta_i, \eta_j, \eta_k, \eta_l\}$, the marginal distribution parameters can be measured using the maximum likelihood [estimation] method by converting $u_i = F(\eta_i)$, $u_j = F(\eta_j)$, $u_k = F(\eta_k)$, and $u_l = F(\eta_l)$. After measuring the copula function parameters using the algorithm described in the preceding section, 1000 intervals are simulated for each of the marginal distributions. The quantile of the portfolio return can be calculated by converting the simulated returns of the marginal distributions to portfolio return. The result is the one-day VaR of the portfolio. This method has been founded upon Jondeau and Rockinger's (2006) paper entitled "XXX" and Huang et al.'s (2009) paper. Both this method and traditional methods and those based on extreme value theory utilize the sliding window procedure. In the first step, data are divided into two groups: in-sample and out-of-sample. Next, the VaR on day 1324 is estimated using the first 1323 observations. For this purpose, the parameters required by the models need to be initially estimated. Then, VaR can be calculated according to the estimated parameters. Afterward, the VaR on day 1325 is estimated based on observations on day 2 until day 1324. This process is continued until the end of the

out-of-sample period to eventually reach 399 estimates for each method at any of the 95% and 99% confidence levels.

4.6. Estimating Value at Risk using traditional methods

To compare the performance of the copula-GARCH model with traditional model, the VaR of the portfolio is calculated by employing traditional methods such as historical simulation and variance-covariance. The historical simulation method assumes that the distribution of returns is repeated. Here, the VaR of the portfolio is calculated using historical data. The variance-covariance method, on the other hand, assumes that the distribution of returns is a normal or Student's t distribution.

4.7. Estimating Value at Risk using the extreme value theory

A) Extreme values:

Extreme values include events like large market falls, failure of enterprises in fulfilling their obligations, financial market crisis, and natural catastrophes. Extreme values form based on the central limit theorem.

B) Generalized Extreme Value (GEV) Theory

Let $f(x)$ be the probability density function, and $F(x)$ be the cumulative distribution function X. Moreover, let the alternating sequence X in periods 1,2, ..., n be represented as X_1, X_2, \dots, X_n . Extreme values are defined as maxima and minima of n independent uniformly distributed random variables X_1, X_2, \dots, X_n . The maximum and minimum values of the variable X during the period n are denoted by $X_{max.n}$ and $X_{min.n}$, respectively:

$$X_{max.n} = \max(X_1, X_2, \dots, X_n) \quad (9)$$

$$X_{min.n} = \min(X_1, X_2, \dots, X_n) \\ = -\max(-X_1, -X_2, \dots, -X_n) \quad (10)$$

where $X_{max.n}$ and $X_{min.n}$ are respectively the maximum and minimum variables obtained from a random sample n. Gumbel (1958) demonstrated that if variables X_1, X_2, \dots, X_n statistically independent and uniformly distributed, the precise distribution of maxima and minima can be reiterated as a function of

parent distribution, i.e., $F(x)$, and the length of the selected period, i.e., n.

$$H_{max.n}(X) = [F(x)]^n \quad (11)$$

where $H_{max.n}(X)$ is the precise distribution of $H_{max.n}$. Jenkinson (X) proposed to represent the generalized distribution of the extreme value simply by the following equation:

$$H_{\epsilon, \mu, \sigma}(x_{max}) = \exp \left\{ - \left[1 + \epsilon_{max} \left(\frac{x_{max} - \mu_{max}}{\sigma_{max}} \right)^{-1/\epsilon_{max}} \right] \right\} \quad (12)$$

where $H_{\epsilon, \mu, \sigma}(x_{max})$ is the cumulative distribution function of the maximum variable.

This distribution includes three parameters, with μ_{max} and σ_{max} as the first and second parameters. The former is the distribution position or criteria parameter, and the latter is the central tendency or scattering measure. These parameters are associated with and at the same time differ from the well-known parameters mean and standard deviation. The third parameter, ϵ_{max} , is the tail index, which implies the shape or density of the distribution sequence.

C) Parameter estimation

Extreme value parameters, namely μ_{max} , σ_{max} , and ϵ_{max} need to be estimated to estimate the risk measures. The desired percentiles can be easily calculated by estimating the above parameters and substituting them in the generalized extreme value (GEV) distribution. The parameters are estimated mainly using the maximum likelihood [estimation] method. This approach has several advantages, including a powerful theoretical foundation, consistent estimators, and asymptotically normal estimators if $\epsilon \geq -1/2$.

D) Calculating generalized extreme value (GEV) distribution percentiles

The GEV distribution percentiles are calculated using the following formula. The resulting value is the GEV distribution percentile for the cumulative probability p.

$$x_{max} = \begin{cases} \mu_{max} - \left(\frac{\sigma_{max}}{\epsilon_{max}} \right) [1 - (-\ln p)^{-\epsilon}] & \text{if (Frechet, } \epsilon > 0) \\ \mu_{max} - \sigma_{max} [\ln(-\ln p)] & \text{if (Gumbel, } \epsilon = 0) \end{cases} \quad (13)$$

E) Calculating Value at Risk (VaR)

To calculate the VaR, the desired percentile must be extracted from the distribution of returns or loss of assets. To transfer the percentiles of the extreme value distribution to those of the parent distribution, a relationship must be established between the probabilities of these distributions in some way. To put it differently, the probability distribution of extreme values X_{max} should be linked to the parent probability distribution X .

$$x = \begin{cases} \text{if(Frechet, } \varepsilon > 0) \\ \mu_{max} - \left(\frac{\sigma_{max}}{\varepsilon_{max}}\right) [1 - (-n \ln(1 - \alpha))^{-\varepsilon_{max}}] \\ \text{if(Gumbel, } \varepsilon = 0) \\ \mu_{max} - \sigma_{max} [\ln(-n \ln(1 - \alpha))] \end{cases} \quad (14)$$

This percentile is the same as the percentage VaR.

Comparing the performance of VaR estimation models

In this study, a novel method for evaluating the performance of VaR models using the fuzzy multicriteria decision-making models is proposed, which has not been addressed in similar research. This section introduces two models for evaluating different VaRs, commonly known as backtesting. In a backtesting method, the accuracy of the VaR model is evaluated using a constant portfolio. Portfolio weights remain constant during backtesting. The results of a given backtesting model depend on portfolio constituents and the VaR model assumptions. This study employs the sliding window method; thus, you need to get familiar with the concept of “hit sequence.” Let’s assume that there are 1500 observations of daily portfolio return in the past, and 1000 observations are used to estimate the VaR. Using the observations on day 1 until day 1000, the VaR on day 1001 is estimated and compared with the actual loss. If the actual loss is greater than the calculated VaR, meaning an exception (success) has occurred, the variable I_{1001} is set to 1 (success); otherwise, it is set to 0 (failure). Then, based on the observations on day 2 until day 1001, the VaR on day 1002 is estimated and compared with the actual loss. If once again an exception has occurred, $I_{1001} = 1$; otherwise, $I_{1002} = 0$. This process is continued until the VaR on day 1500 is estimated based on observations on day 500 until day 1499. Then, I_{1500} is calculated. Now, we have an $n = 500$

series of 0 and 1, referred to as “hit sequence,” which indicates whether VaR has been violated in the last 500 days. In other words, if $X_{t,t+1}$ denotes actual loss from day t until day $t + 1$, the hit function can be expressed as follows:

$$I_{t+1}(\alpha) = \begin{cases} 1 & \text{if } X_{t,t+1} \leq -VaR(\alpha) \\ 0 & \text{if } X_{t,t+1} > -VaR(\alpha) \end{cases} \quad (15)$$

Therefore, a hit sequence like (1, 00, 1, 0, ..., 1) determines whether VaR has been violated in the past.

1) Unconditional coverage property

A loss event probability greater than the reported VaR, $VaRt(\alpha)$ must be precisely equal to $\alpha \times 100\%$ (i.e., $pr(I_{t+1}(\alpha) = 1) = \alpha$). The observation of losses higher than VaR (successes) with a repetition greater than $\alpha \times 100\%$ suggests that the VaR criterion systematically underestimates the actual risk of the portfolio. Below, two different backtesting methods will be introduced.

Unconditional coverage test

The unconditional coverage test, which was introduced by Kupiec (1995), is conducted based on the number of successes. It was later generalized by Christoffersen (1998) to the conditional state. The unconditional coverage test indeed tests the null hypothesis that the hit function (which is assumed to have independent, identically distributed (i.i.d.) Bernoulli distribution) has a constant probability of success equal to the significance level (α) of the calculated VaR model. The test statistic is as follows:

$$LRuc = \frac{\pi_{exp}^{n_1} (1 - \pi_{exp})^{n_0}}{\pi_{obs}^{n_1} (1 - \pi_{obs})^{n_0}} \quad (16)$$

where n is the total sample size, n_1 is the total number of successes, and $n_0 = n - n_1$ is the [total] number of failures. Moreover, $\pi_{exp} = \alpha$ and $\pi_{obs} = n_1/n$. The result of multiplying the natural logarithm of this statistic by -2 will be asymptotically chi-square distributed. If the value of the calculated statistic is greater than the critical value, the null hypothesis is rejected, meaning that the VaR estimation model is not accurate.

Dowd’s ranking method

Dow proposed a novel method for comparing various VaR estimation models. Here, a loss function is first defined as follows:

$$C_t = \begin{cases} (VAR_{\alpha,t} - r_t) & \text{if } r_t < VAR_{\alpha,t} \\ 0 & \text{if } r_t > VAR_{\alpha,t} \end{cases} \quad (17)$$

The quadratic score function can be expressed as follows. The method with the least loss is the best method.

$$QS = \frac{2}{n} \sum_{t=1}^n (C_t)^2 \quad (18)$$

Comparing performance using fuzzy TOPSIS

This study uses the ranks obtained from the VaR estimation models based on the unconditional coverage test procedure (as a statistical method that

has caught the attention of risk-indifferent investors), the Dowd’s loss function method (as an important method from risk-averse (conservative) investors’ point of view), and ranking given the prediction accuracy level (as an important method from risk-taking investors’ point of view), as the decision-making matrix of the multicriteria decision-making model. The fuzzy multicriteria decision-making (fuzzy TOPSIS) method is employed because the decision-maker has attached relative importance to the three methods above. In Chen and Huang’s fuzzy TOPSIS method, the following steps are generally taken: after running the algorithm, the decision-making matrix *D* (an *m* * *n* matrix) and the index weights vector relative to the target *W*, as the input of the algorithm, must be initially formed:

$$D = \begin{matrix} & X_1 & \dots & X_j & \dots & X_n \\ \begin{matrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} \tilde{x}_{11} & \dots & \tilde{x}_{1j} & \dots & \tilde{x}_{1n} \\ \vdots & & \vdots & & \vdots \\ \tilde{x}_{i1} & \dots & \tilde{x}_{ij} & \dots & \tilde{x}_{in} \\ \vdots & & \vdots & & \vdots \\ \tilde{x}_{m1} & \dots & \tilde{x}_{mj} & \dots & \tilde{x}_{mn} \end{bmatrix} \end{matrix}$$

$$\begin{cases} \tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}) & \text{If it is a triangular fuzzy number.} \\ \tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}) & \text{If it is a trapezoidal fuzzy number.} \end{cases}$$

$$W = (\tilde{w}_1, \dots, \tilde{w}_j, \dots, \tilde{w}_n); \begin{cases} \tilde{w}_j = (\alpha_j, \beta_j, \chi_j) & \text{If it is a triangular fuzzy number.} \\ \tilde{w}_j = (\alpha_j, \beta_j, \chi_j, \delta_j) & \text{If it is a trapezoidal fuzzy number.} \end{cases} \quad (19)$$

Step 1. Normalizing the decision-making matrix. In the first step, the decision-making matrix should be normalized to make its elements “unscaled.” There are several normalization methods. For example, Chen and Huang applied the linear normalization method. For this purpose, the maximum value of each column *x⁺_j* and the minimum value of each column *x_j⁻* need to be determined. Furthermore, *r_{ij}* values can be calculated using the following equations:

When *X_{ij}*’s are fuzzy (a triangular number: *x_{ij} = (a_{ij}, b_{ij}, c_{ij})* or a trapezoidal number), *r_{ij}*’s are definitely fuzzy as well. If fuzzy numbers are triangular and *x_j⁺ = (a_j⁺, b_j⁺, c_j⁺)* and *x_j⁻ = (a_j⁻, b_j⁻, c_j⁻)* are respectively the maximum and minimum scores, then we have:

$$\tilde{r}_{ij} = \begin{cases} \tilde{x}_{ij}(/)\tilde{x}_j^+ = \left(\frac{a_{ij}}{c_j^+}, \frac{b_{ij}}{b_j^+}, \frac{c_{ij}}{a_j^+}\right); & \text{If } \tilde{x}_{ij} \text{ has a positive aspect.} \\ \tilde{x}_j^-(/)\tilde{x}_{ij} = \left(\frac{a_j^-}{c_{ij}}, \frac{b_j^-}{b_{ij}}, \frac{c_j^-}{a_{ij}}\right); & \text{If } \tilde{x}_{ij} \text{ has a negative aspect.} \end{cases} \quad (20)$$

Thus, the normalized matrix *D* is reduced (converted) to *D’*:

$$D' = \begin{matrix} & X_1 & \cdots & X_j & \cdots & X_n \\ \begin{matrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} \tilde{r}_{11} & \cdots & \tilde{r}_{1j} & \cdots & \tilde{r}_{1n} \\ \vdots & & \vdots & & \vdots \\ \tilde{r}_{i1} & \cdots & \tilde{r}_{ij} & \cdots & \tilde{r}_{in} \\ \vdots & & \vdots & & \vdots \\ \tilde{r}_{m1} & \cdots & \tilde{r}_{mj} & \cdots & \tilde{r}_{mn} \end{bmatrix} \end{matrix} \quad (21)$$

Step 2. Obtaining the normalized weighted matrix.

For triangular fuzzy numbers:

$$\begin{cases} \tilde{v}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j = \left(\frac{a_{ij}}{c_j^+}, \frac{b_{ij}}{b_j^+}, \frac{c_{ij}}{a_j^+}\right)(\cdot)(\alpha_j, \beta_j, \chi_j) = \left(\frac{a_{ij}}{c_j^+} \cdot \alpha_j, \frac{b_{ij}}{b_j^+} \cdot \beta_j, \frac{c_{ij}}{a_j^+} \cdot \chi_j\right) \\ \tilde{v}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j = \left(\frac{a_j^-}{c_{ij}}, \frac{b_j^-}{b_{ij}}, \frac{c_j^-}{a_{ij}}\right)(\cdot)(\alpha_j, \beta_j, \chi_j) = \left(\frac{a_j^-}{c_{ij}} \cdot \alpha_j, \frac{b_j^-}{b_{ij}} \cdot \beta_j, \frac{c_j^-}{a_{ij}} \cdot \chi_j\right) \end{cases} \quad (22)$$

For trapezoidal fuzzy numbers:

$$\begin{cases} \tilde{v}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j = \left(\frac{a_{ij}}{d_j^+}, \frac{b_{ij}}{c_j^+}, \frac{c_{ij}}{b_j^+}, \frac{d_{ij}}{a_j^+}\right)(\cdot)(\alpha_j, \beta_j, \chi_j, \delta_j) = \left(\frac{a_{ij}}{d_j^+} \cdot \alpha_j, \frac{b_{ij}}{c_j^+} \cdot \beta_j, \frac{c_{ij}}{b_j^+} \cdot \chi_j, \frac{d_{ij}}{a_j^+} \cdot \delta_j\right) \\ \tilde{v}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j = \left(\frac{a_j^-}{d_{ij}}, \frac{b_j^-}{c_{ij}}, \frac{c_j^-}{b_{ij}}, \frac{d_j^-}{a_{ij}}\right)(\cdot)(\alpha_j, \beta_j, \chi_j, \delta_j) = \left(\frac{a_j^-}{d_{ij}} \cdot \alpha_j, \frac{b_j^-}{c_{ij}} \cdot \beta_j, \frac{c_j^-}{b_{ij}} \cdot \chi_j, \frac{d_j^-}{a_{ij}} \cdot \delta_j\right) \end{cases} \quad (23)$$

The first equation is applied when the *j*th criterion has a positive aspect, while the second equation is applied when the *j*th criterion has a negative aspect. The calculation results are then inserted into the matrix *v*, yielding the following matrix:

$$v = \begin{matrix} & X_1 & \cdots & X_j & \cdots & X_n \\ \begin{matrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} \tilde{v}_{11} & \cdots & \tilde{v}_{1j} & \cdots & \tilde{v}_{1n} \\ \vdots & & \vdots & & \vdots \\ \tilde{v}_{i1} & \cdots & \tilde{v}_{ij} & \cdots & \tilde{v}_{in} \\ \vdots & & \vdots & & \vdots \\ \tilde{v}_{m1} & \cdots & \tilde{v}_{mj} & \cdots & \tilde{v}_{mn} \end{bmatrix} \end{matrix} \quad (24)$$

Step 3. Finding the positive ideal solution (PIS), denoted by A^* and negative ideal solution (NIS), denoted by A^- .

where $\tilde{v}_j^* = \max_i\{v_{ij}\}$, $\tilde{v}_j^- = \min_i\{v_{ij}\}$, $A^* = (\tilde{v}_1^*, \tilde{v}_2^*, \dots, \tilde{v}_n^*)$ (25)
 $i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n. \quad A^- = (\tilde{v}_1^-, \tilde{v}_2^-, \dots, \tilde{v}_n^-)$

Step 4. Measuring the distance of each option from the positive and negative ideal (d_i^+, d_i^-).

$$\begin{aligned} d_i^+ &= \sum_{j=1}^n d_v(\tilde{v}_{ij}, \tilde{v}_j^*), \quad i = 1, 2, \dots, m \\ d_i^- &= \sum_{j=1}^n d_v(\tilde{v}_{ij}, \tilde{v}_j^-), \quad i = 1, 2, \dots, m \end{aligned} \quad (26)$$

where $d_v(0,0)$ is the distance measurement between two fuzzy numbers.

Step 5. Calculating the relative proximity of each option to the ideals (C_i^+). This index is defined to combine the S_i^+ and S_i^- values, and consequently, to compare the options, which can be calculated using the following equation:

$$C_i^+ = \frac{S_i^+}{S_i^+ + S_i^-} \quad (27)$$

Step 6. Ranking the options. The options can be ranked based on the descending order of C_i^+ 's values.

5. Findings

Descriptive statistics of research data

In the sample under study, each asset has 1723 daily returns over a seven-year period. Table 1 presents the details of one symbol (وبانک).

Table 1. Descriptive statistics of research data

	Investment company (وبانک)	
Sample size	1722	
Mean	0.00177	
SD	0.0099	
Skewness	0.3563	
Kurtosis	4.1347	
ARCH test	Q-statistics	P-value
LM(4)	250.2639	0
LM(6)	268.8194	0
LM(8)	269.4590	0
LM(10)	281.9260	0
Jarque-Bera test	JB-statistics	P-value
	128.81	0.000

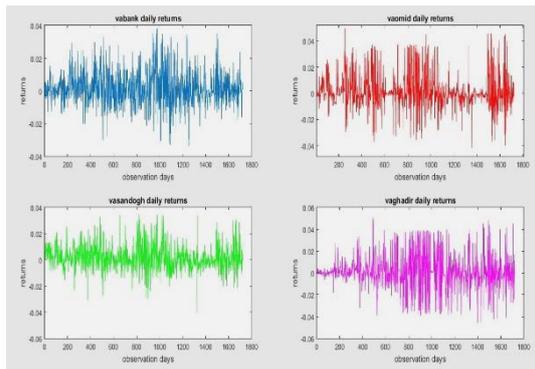


Fig. 1: Daily return of assets in this study

Fig. 1 clearly shows the effect of ARCH on four time-series of asset returns. Hence, it was tested using the ARCH test, the results of which are presented in Table 1. The ARCH test results suggest that ARCH affects all the four time-series, indicating heteroscedasticity in the time-series. Furthermore, all the four time-series have large kurtosis, suggesting that the distribution of asset returns has a longer sequence than the normal distribution. The Jarque-Bera test is utilized to test the normality of the assets introduced in this research. The results (Tables 1 and 2) imply their non-normality.

Results of marginal models (explaining and estimating the mean and conditional variance model)

To achieve i.i.d. distributions and to estimate the variance, the mean and conditional variance model is

simultaneously explained and estimated here. To this aim, the Box-Jenkins methodology (ARIMA) is applied to model the average return of assets in this research and heteroscedasticity models (GARCH) to model the research conditional variance. The Box-Jenkins method seeks to answer the following question, “How can we determine whether a given time-series follows a pure autoregressive process, a pure moving average process, or an autoregressive-moving average (ARMA) process.

Estimating the Value at Risk using the copula-ARIMA-GARCH model

In the first step, a model is estimated for each marginal distribution using the ARIMA-GARCH model. This is followed by the standardization of the error terms in these models. If the standardized error term time-series of the marginal distributions is demonstrated as $\{\eta_i, \eta_j, \eta_k, \eta_l\}$, the marginal distribution parameters can be measured using the maximum likelihood [estimation] method by converting $u_i = F(\eta_i)$, $u_j = F(\eta_j)$, $u_k = F(\eta_k)$, and $u_l = F(\eta_l)$. After measuring the copula function parameters using the algorithm described in Chapter 3, 1000 intervals are simulated for each marginal distribution. The quantile of the portfolio return can be calculated by converting the simulated returns of the marginal distributions to portfolio return $X_{p,t} = w_1X_{i,t} + w_2X_{j,t} + w_3X_{k,t} + w_4X_{l,t}$.

Comparing the various methods

For a statistical test, the already explained backtest methods are used. Table 8 shows Kupiec’s unconditional coverage test results. Given the calculated statistic values at 5% and 1% confidence levels and their corresponding critical values, it is observed that the null hypothesis is rejected at 95% and 99% confidence levels for the variance-covariance method. It has not been rejected for other methods, indicating the inaccuracy of the variance-covariance model.

Table 3 shows the results of Dowd’s ranking method and the ranks assigned to the methods at the significance level of VaR calculation. Naturally, the results of the variance-covariance model are not presented in subsequent tables because it has been rejected in the unconditional coverage test. Due to the very slight difference between the estimated VaRs and the actual portfolio return, this criterion seems to have

failed in accurately rank the different VaRs, as the results are quite close to each other.

Table 2. Unconditional coverage test results

Trading days	399	
Critical value 5%	3.84	
Critical value 1%	6.63	
VaR estimation method	5% statistic	1% statistic
copula-ARIMA-GARCH model	0.2071	0.000025
Historical simulation model	2.8914	0.2714
Variance-Covariance model	212.17	337.72
GEV model	2.5586	0.8859

Table 3. Results of Dowd’s ranking method at 5% and 1% confidence levels

Significance level	5%		1%	
	Dowd’s loss function	Rank	Dowd’s loss function	Rank
VaR estimation method				
copula-ARIMA-GARCH model	0.000933795	2	0.001539425	3
Historical simulation model	0.001128313	3	0.000940092	1
GEV model	0.000920131	1	0.0010694	2

Comparing the performance using fuzzy TOPSIS

This study uses the ranks obtained from the VaR estimation models based on the unconditional coverage test procedure (as a statistical method that has caught the attention of risk-indifferent investors), the Dowd’s loss function method (as an important method from risk-averse (conservative) investors’ point of view), and ranking given the prediction accuracy level (as an important method from risk-taking investors’ point of view), as the decision-making matrix of the multicriteria decision-making model. The fuzzy multicriteria decision-making (fuzzy

TOPSIS) method is employed because the decision-maker has attached relative importance to the three methods above. This study utilizes Chen’s (2006) fuzzy TOPSIS method, thanks to the wide application of the TOPSIS method and its advantages over other MCDM methods. To determine the degree to which risk-aversion (or risk-taking) affects the decision-making process, we intend to define a variety of investors and decision-makers.

In this study, Chen and Huang’s scale is applied to convert linguistic terms to fuzzy numbers, as illustrated in Fig. 2.

Table 4. Classification of various decision-makers and financial institutions

The relative utility of the statistical method	The relative utility of the prediction accuracy	The relative utility of the time function	Type of investor
Very low	Low	High	Risk-averse
Very low	Medium	Medium	Risk-indifferent
Very low	High	Low	Risk-taking

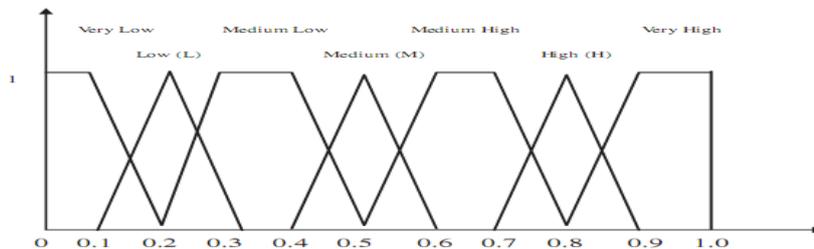


Fig. 2: Chen and Huang’s scale to convert linguistic terms to fuzzy numbers

This method utilizes linear normalization to simplify (streamline) calculations, along with triangular fuzzy numbers. Finally, the decision-making and ranking processes are conducted by computing the closeness coefficient (CC_i) for the i th alternative. The ranks obtained from each estimate using the unconditional coverage test procedure, Dowd's loss function method at 1% and 5% confidence levels and ranking based on prediction accuracy are displayed in Table 5. This is indeed our decision-making matrix, which is the primary input of our decision-making methods.

The initial normalized decision-making matrix is multiplied by the relative weight of the criteria. Then, the distance of the alternatives from these two points is calculated by defining the positive and negative fuzzy ideals. Next, the closeness coefficient is measured by dividing the distance of each alternative from the

negative ideal by the sum of the distances of the alternative from the ideals. This coefficient is used to rank the estimators and for decision-making. The normalized decision-making matrix is presented in Table 6.

For example, Table 7 depicts the results of multiplying the normalized decision-making matrix by the relative weight of criteria for a risk-averse investor. Using the results shown in Table 7, the fuzzy positive ideal solution (FPIS) and the fuzzy negative ideal solution (FNIS) are defined as follows.

After calculating the distance of the alternatives from the positive and negative ideal points, the closeness coefficient (CC) is calculated, whose results are presented in Table 8.

Then, estimator ranking and decision-making are performed based on the closeness coefficient results and Table 4.

Table 5. Initial decision-making matrix

Indices Alternatives	Obtained rank based on the 5% loss function	Obtained rank based on the 1% loss function	Obtained rank based on the prediction accuracy	Obtained rank based on the unconditional coverage method
copula-ARIMA-GARCH model	2	3	1	1
Historical simulation model	3	1	3	1
GEV model	1	2	2	1
Variance-covariance model	4	4	4	2

Table 6. Initial normalized decision-making matrix

Indices Alternatives	Obtained rank based on the 5% loss function	Obtained rank based on the 1% loss function	Obtained rank based on the prediction accuracy	Obtained rank based on the unconditional coverage method
copula-ARIMA-GARCH model	0.5	0.33	1	1
Historical simulation model	0.33	1	0.33	1
GEV model	1	0.5	0.5	1
Variance-covariance model	0.25	0.25	0.25	0.5

Table 7. Product of the normalized decision-making matrix by the relative weight of criteria for a risk-averse investor

Indices Alternatives	Obtained rank based on the 5% loss function	Obtained rank based on the 1% loss function	Obtained rank based on the prediction accuracy	Obtained rank based on the unconditional coverage method
copula-ARIMA-GARCH model	(0.05, 0.1, 0.15)	(0.03, 0.06, 0.1)	(0.7, 0.8, 0.9)	(0.0, 0.0, 0.2)
Historical simulation model	(0.03, 0.06, 0.1)	(0.1, 0.2, 0.3)	(0.23, 0.26, 0.3)	(0.0, 0.0, 0.2)
GEV model	(0.1, 0.2, 0.3)	(0.05, 0.1, 0.15)	(0.35, 0.4, 0.45)	(0.0, 0.0, 0.2)
Variance-covariance model	(0.025, 0.05, 0.075)	(0.025, 0.05, 0.075)	(0.175, 0.2, 0.225)	(0.0, 0.0, 0.2)

Table 8. Closeness coefficient for a risk-averse investor

Closeness coefficient Alternatives	Closeness coefficient (CC)
copula-ARIMA-GARCH model	0.72
Historical simulation model	0.26
GEV model	0.45
Variance-covariance model	0.00

Ranking the estimators for a risk-averse investor

copula-ARIMA-GARCH>GEV>Historical simulation>variance-covariance model

The preceding process is repeated for risk-indifferent and highly risk-taking investors. The results of closeness coefficients are shown in Table 9.

Table 9. Closeness coefficient for risk-taking and risk-indifferent investors

Closeness coefficient Alternatives	Closeness coefficient (CC) for a risk-taking investor	Closeness coefficient (CC) for a risk-indifferent investor
copula-ARIMA-GARCH model	0.39	0.61
Historical simulation model	0.46	0.34
GEV model	0.60	0.51
Variance-covariance model	0.00	0.00

Ranking the estimators for a risk-taking investor

GEV>Historicalsimulation>copula-ARIMA-GARCH>Variance-covariance model

Ranking the estimators for a risk-taking investor

copula-ARIMA-GARCH>GEV>Historical simulation>Variance-covariance model

6. Conclusions

Investors take into consideration the project risk and return at the same time as one of the primary factors contributing to investment decisions when they are about to invest in different projects. It should be acknowledged that risk is an integral part of the return, which must be taken into account when deciding the return of different investment projects.

This study aimed to calculate the risk more accurately and develop methods of performance evaluation of value at risk (VaR) models. Herein, by building a portfolio composed of the shares of four investment companies, with stock symbols A, B, C, and D we seek to answer assumption study. Copula functions used to calculate the VaR of the portfolio in the field of risk management. The results indicated that combining the copula theory with the ARIMA-GARCH model could serve as a powerful tool for calculating the VaR. The mean and conditional variance model was simultaneously explained and estimated to obtain i.i.d. distributions and to estimate

the variance. For this purpose, the Box-Jenkins methodology (ARIMA) was employed to model the average return of assets in the research and heteroscedasticity models (GARCH) to model the research conditional variance. Herein, a novel method was proposed to evaluate the performance of VaR models using the fuzzy multicriteria decision-making (MCDM) models. This study uses the ranks obtained from the VaR estimation models based on the unconditional coverage test procedure (as a statistical method that has caught the attention of risk-indifferent investors), the Dowd’s loss function method (as an important method from risk-averse (conservative) investors’ point of view), and ranking given the prediction accuracy level (as an important method from risk-taking investors’ point of view), as the decision-making matrix of the multicriteria decision-making model. The fuzzy multicriteria decision-making (fuzzy TOPSIS) method is employed because the decision-maker has attached relative importance to the three methods above. This study utilizes Chen’s (2005) fuzzy TOPSIS method, thanks to the wide application of the TOPSIS method and its advantages over other MCDM methods. The results revealed that the calculated VaR by the copula-ARIMA-GARCH model had yielded the best performance in risk-taking and risk-indifferent decision-makers and investors’ opinion due to the importance they attach to the

prediction accuracy. Besides, risk-averse investors consider the GEV model as the best method due to the importance they attach to the loss function.

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