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Robust optimization for identifying the most efficient decision-making unit in data envelopment analysis

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Abstract

Due to the nonlinear and discrete nature of BCC (Banker, Charnes, and Cooper, [1]) models for determining the most efficient decision-making unit, it is practically impossible to evaluate the models' dual and, consequently, optimistic case. Thus, in this paper, the linear model with linear constraints proposed by Akhlaghi et al. [2] is used to investigate the dual equality of the model's robust problem and the optimistic case of the new model's dual under VRS uncertainty. The model proposed in this paper is novel in comparison to previous models because it solves the most efficient decision-making unit only once, without relying on uncertain data to determine its rank. The paper demonstrates how the proposed robust model can also ascertain the most efficient decision-making unit when uncertainty exists. Furthermore, the dual issues raised by robust counterparts in the new linear programming (LP) model are addressed to identify the most efficient decision-making unit. The robust counterpart is demonstrated to be equivalent to a linear program under interval uncertainty, and the dual of the robust counterpart is shown to be equal to the optimistic counterpart of the dual problem. Consequently, this study aims to demonstrate that the dual problem is equivalent to a decision-maker operating under optimal data, whereas the primal robust problem is equivalent to a decision-maker operating through the worst-case possible data scenario.

Keywords: Robust Optimization, Optimistic Counterpart, Uncertainty, Data Envelopment Analysis (DEA), Interval Data.

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1. Introduction

Data Envelopment Analysis (DEA) is a popular optimization technique for determining the relative effectiveness of a group of homogeneous Decision-Making Units (DMUs) [3]. The primary objective of certain methods is to select the most efficient DMU rather than to rank DMUs. Therefore, it appears unnecessary to evaluate the performance of each DMU in this condition [4-7]. Consequently, it is essential to directly introduce the most efficient DMU [1,4,8,9]. Small changes in the data have been shown to significantly affect the nominal optimal solution and its feasibility and optimality. This implies that the solution may become meaningless; hence, a practical optimization problem should be modeled and considered an uncertain data problem [10-12]. When there is uncertainty in the data used to determine the solution, there will be some ambiguity regarding the exact values of particular data components. However, in DEA, some data are inherently uncertain. The authors propose several techniques for addressing optimization issues in the presence of uncertainty and perturbed optimization problems, one of which is robust optimization, which has recently gained considerable attention [13,14]. When the underlying data for efficiency analysis is unknown, which is an inevitable feature in real-world scenarios, the classification of DMUs as either efficient or not could be quite misleading. For example, in the financial or aviation industry, accurate performance evaluation is crucial for revenue improvement, as a result, any uncertainty in data could affect management decisions towards the right potential improvement of a DMU or revenue [15,16]. Given that uncertainty a feature of any business environment, and its effect as such is dismissive of performance evaluation, it has become imperative to consider uncertain parameters in the evaluation process and improve the robustness of DEA. Over the years, the issue of robustness, aiming at solutions that are stable to parameter perturbation and preserving the efficiency of DMUs, has been the subject of extensive research by many re-searchers. Research in this direction has been significant because of how robust analysis accounts for uncertainties observed in real-world decision problems [15]. Specifically, three major reasons explain why considering uncertainty and ensuing robust analysis is crucial in DEA. Firstly, the efficiency scores derived for each DMU are obtained by comparing each unit to the other. Therefore, uncertain data could lead to incorrect reference units or the selection of the incorrect best practice unit. In fact, the efficiency or inefficiency of a DMU could be brought to question by a small uncertainty in data. As emphasized in Ehrgott et al. [17], there is a reasonable argument against the perceived performance of a DMU when the underlying data is imperfect. Secondly, the DEA measures the improvement of the inefficient unit relative to the efficient frontier. This implies that the right amount of potential improvement needed to project an inefficient unit to an efficient one becomes difficult to be measured. Thirdly, the DEA model fails to preserve the efficiency of the DMUs since the model becomes sensitive to a small perturbation in the underlying parameters and data. Several robust techniques have been proposed in the literature to overcome the issue of uncertainty in data.

One of the earliest considerations of uncertainty in the mathematical programming community is the use of sensitivity analysis. In DEA, sensitivity analysis dates back to the work of Charnes et al [1] and focuses on ensuring the stability of the classification of DMUs into efficient and inefficient through preserving the efficiency of DMUs. Algorithmic and distance defining metric techniques are applied to solve this problem, which includes defining a stability region for which data variations will not change a DMU’s classification or omitting an efficient DMU and consequently, changing the reference set for the DMU [18,19]. However, sensitivity analysis is only a post-efficiency analysis because it measures how the efficiency scores with respect to data variation or observations differ from their actual efficiency scores. Therefore, they are not quite an effective measure for robustness [12]. In recent times, stochastic and deterministic approaches that deal with uncertainty in DEA data from the onset and incorporate expert opinions under uncertain environments have been introduced [17,20]. The degree to which the efficiency of DMUs is stable to the underlying uncertainties in the input and output data usually reflects the robustness of the DEA model. The relative efficiency of a DMU is regarded robust when all input and output weights or their representative sample in the DEA models are feasible with respect to its uncertain or imprecise data [15]. Robustness in DEA can be measured with several approaches as mentioned in the previous section. This paper adopts the concept of robustness offered through the lens of robust optimization and applied it to the DEA, known as the RDEA. The robust optimization technique was introduced in Soyster [20] and developed by Mulvey et al. [21], Ben-Tal and Nemirovski (1998) [12], and Bertsimas & Sim [22] among others. The obtained solution of the robust optimization exhibit stability and can withstand changes in the parameters of the model with-out affecting the solution. The reader is referred to Gorissen et al. [23] for a practical guide to robust optimization. Loosely speaking, a solution under the robust optimization is said to be robust if it is obtained through a robust counterpart an alternative formulation of the nominal optimization problem, that seeks all or most possible realization of the uncertain parameters in an uncertainty set defined by the user. Like robust optimization, the RDEA seeks to similarly immunize uncertain inputs and outputs parameters in a user-defined uncertainty set and provides a probability guarantee for constraint feasibility and reliable performance evaluation, and stable classification of DMUs. Thus, an efficiency of a DMU is said to be ‘robust’ if it’s uncertain inputs and outputs parameters are immunized in an uncertainty set, feasible with respect to the un-certain data and the efficiency is stable to data perturbation. The RDEA method results in an efficiency that is near-optimal efficiency of the nominal DEA and requires no knowledge of the probability distribution of the uncertain input and output data. The RDEA was originally initiated by Sadjadi and Omrani [24] The authors assumed the existence of uncertainty in output data of DMUs and adapted the robust optimization approach of Ben-Tal and Nemirovski [12] and Bertsimas & Sim [22] to correct the efficiency of DMUs. Such robust concept, although similar in objective to the aforementioned approaches (i.e., the statistical-based robust non-parametric estimation methods applied in the work of Cazals et al. [25] and Daraio and Simar [27,28], the CCDEA of [26]) and IDEA of Cooper, Park, & Yu [27] is quite different due to its immunity against noise, uncertainty, and flexibility in obtaining the robust efficiency. The successful and wide applications of the RDEA method are documented in Peykani et al. [28]. There have been theoretical and practical extensions of the RDEA since the initial model of Sadjadi and Omrani [24]. In Sadjadi and Omrani [29], the authors combined the RDEA and boot-strapping technique to measure the efficiency of telecommunication companies. Arabmaldar et al. [30] proposed an RDEA model and robust super-efficiency DEA measures under constant returns to scale (CRS) technology whereas Salahi et al. [31] proposed a robust Russell measure under interval and ellipsoidal uncertainties in their best and worst-cases. Toloo & Mensah [32] studied non-negativity conditions in robust optimization and proposed a reduced robust DEA based on variable returns scale (VRS) technology. They applied their model to the efficiency of the largest banks in Europe and showed the computational advantage of the model. Recently, Tavana et al. [33] developed two DEA adaptations to rank DMUs characterized by interval data and undesirable outputs. They applied their model to assess cross-efficiency and real-life bank data. Considering similar adaptation of interval data and non-discretionary factors, Arabmaldar et al. [35] proposed a robust worst-practice model to the worst-performing suppliers where some factors in the supply chain decision analysis are not under the discretion of management. Hatami-marbini and Arabmaldar [35] extended the RDEA to estimate Farrell's cost efficiency in situations of endogenous and exogenous uncertainties. In the endogenous case, uncertainty in input and output data is assumed whereas exogenous uncertainty is considered for prices of inputs. In the latter case, the robust DEA estimating lower and upper bound of cost efficiencies is given. Although several studies have been done on the RDEA, it still needs further research and development on the methodology.

The main aim of this paper is two-fold: Firstly, First, we showed that in all the previous models, the most efficient unit is proposed in the deterministic space [4,18,36,37] And considering that many events are uncertain, using the efficient model of the robust most efficient unit can give us a more reliable result in the case of uncertainty.

2. literature review

During the last two decades, data envelopment analysis (DEA) has been widely utilized in many operations research. Karsak and Ahiska [38] proposed model for finding the most efficient DMUwith a single input and s outputs, Foroughi [5] proposed two-stage approach to find the most efficient unit and also to fully rank all the DMUs, Toloo& Kresta, [39] find the most CW-efficient DMU when there are no explicit outputs, Wang and Jiang [40] proposed MILP model under constant returns to scale

(CRS), All the mentioned models are widely used in the deterministic space and some of them are non-linear, but in the real world some phenomena are in the non-deterministic space, for this reason we use the Robust model. Salahi, Torabi, and Amiri (2016) developed a robust counterpart model for the CCR model and established relations between CCR's dual robust counterpart and optimistic robust counterpart. Toloo and Mensah (2018) proposed alternate robust counterparts for nonnegative decision variables. Tavana et al. [33] developed two DEA adaptations to rank DMUs characterized by interval data and undesir- able outputs. They applied their model to assess cross-efficiency and real-life bank data. Considering similar adaptation of interval data and non-discretionary factors, Arabmaldar et al. [34] proposed a robust worst practice model to the worst-performing suppliers where some factors in the supply chain decision analysis are not under the discretion of management. Robust DEA (henceforth RDEA) is the application of RO in DEA. The first application of the robust optimization to DEA began in 2008 with Sadjadi & Omrani [24] when they investigated the performance of utility service providers where the underlying data was uncertain. The authors focused on providing a robust and reliable performance ranking of DMUs for management decision in the utility service. Furthermore, the work of Sadjadi et al [24], Wang et al [40] bolstered the need for robust efficiency measure via the RO. For more information, readers can see [41].

3. Robust DEA

Sadjadi & Omrani [30] were the pioneer researchers that worked on the RDEA model with consideration of uncertainty on output parameters for measuring the performance of Iranian electricity distribution companies. They proposed the robust CCR model based on the robust approaches of Ben-Tal, & Nemirovski. [12,14] Based on Sadjadi, & Omrani [29] study, for considering the uncertainty on outputs, the conventional CCR model that is presented Charnes et all [1] is transformed into (1)

$$
\begin{aligned}
\max \quad & \theta \qquad\qquad (1) \\
& \sum_{r=1}^{s} u_r \tilde{y}_{ro} > \theta \\
s.t \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad \forall j \\
& \sum_{i=1}^{m} v_i x_{io} = 1 \\
& v_i, u_r \ge 0
\end{aligned}
$$

Note that each entry $\tilde{y}$ is determined as a symmetric and bounded random variable, which takes the values $[y - \hat{y}.y + \hat{y}]$, where the center of this interval at the point y is a nominal value and $\hat{y}$ is the perturbation of uncertain parameters $\hat{y}$. The RDEA model according to robust optimization formulation of Ben-Tal, & Nemirovski. [12] is proposed as (2)

$$\max C^T x \qquad (2)$$

$$\begin{aligned}
s.t \quad & \sum_j a_{ij}x_j + z_i\tau_i + \sum_{j\in J_i} p_{ij} \le b_i \; \forall i \\
& z_i + p_{ij} \ge \hat{a}_{ij} y_j, \quad \forall i \ne 0, j \in J_i \\
& p_{ij} \ge 0, \quad \forall i, j \in J_i \\
& y_j \ge 0, \quad \forall j \\
& z_i \ge 0, \quad \forall i \\
& -y_j \le x_j \le y_j \quad \forall j \\
& -l_j \le x_j \le u_j \quad \forall j
\end{aligned}$$

As is seen in Model (1,2), to propose RDEA model for dealing with continuous uncertain data, the robust counterpart of all uncertain constraints should be written based on one of the convex uncertainty sets. With respect to this fact that decision maker (DM) can adjust the robustness and linearity of the RDEA model founded upon the Bertsimas & Sim (2004)'s RO method, this model is more effective and applicable in the literature of robust DEA literature.

4. Best efficient unit under uncertainty

This study in deterministic space proposes the continuous linear model proposed by Akhlaghi et al. [2], which is practically superior to other nonlinear models. Applying chaos/disorder to interval data with varying epsilons revealed that optimality remains constant. Initially, we provide a robust counterpart to the LP model shown below to identify the most efficient unit utilizing VRS technology. This is accomplished by replacing binary variables θ_j with a continuous range $0 \le \theta_j \le 1$ for $j = 1,...,n$:

Where x_{ij} denotes the i-th input value for j-th DMU, y_{rj} is the r-th output value for j-th DMU,

u_r represents the weight values of the r-th output, v_i denotes the weight values of the i-th input,

and ε is a non-Archimedean infinitesimal number. by Akhlaghi et al. [2] proposed the most efficient model in deterministic space as follows:

$$\min d_{\max} \qquad (3)$$

$$\begin{aligned}
s.t \quad & \sum_{i=1}^{m} v_i x_{ij} \le 1, & j = 1,...,n \\
& \sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0, & j = 1,...,n \\
& d_{\max} - d_j \ge 0, & j = 1,...,n \\
& \sum_{j=1}^{n} \theta_j = n-1, & j = 1,...,n \\
& 0 \le d_j \le \theta_j \le 1, & j = 1,...,n \\
& \theta_j \le N d_j, & j = 1,...,n \\
& v_i \ge \varepsilon^*, & i = 1,...,m \\
& u_r \ge \varepsilon^*, & r = 1,...,s \\
& d_{\max} \ge 0 & \\
& u_0 \text{ is free} &
\end{aligned}$$

The models' optimistic counterparts can also be used to determine the most efficient decision-making unit (DMU) when input and output parameters are uncertain. The efficient decision-making unit is determined optimistically and pessimistically for the best- and worst-case scenarios. Several critical concepts are defined as follows:

Definition 1: Robust counterpart optimization is a problem that seeks to minimize the robust target function for all robust solutions to a given set of uncertainties.

The following is a generalization of the uncertain optimization problem.

$$\begin{aligned} \min_x \quad & g(x,u) \qquad (4) \\ s.t \quad & f_i(x,v_i) \le 0, \quad i=1,\dots,m \\ & x \in R^n \end{aligned}$$

In this equation, g and f_i denotes the variable's x convex functions, $u \in R^p$ and $v_i \in R^{q_i}$ represents the problem's uncertain but bounded parameters, U and V_i represent uncertainty sets in which $u \in U$ and $v_i \in V_i, (i=1,\dots,m)$ and x is the feasible solution to the problem (4). If constraints exist on the uncertain parameter $v_i \in V_i$, the robust counterpart of the problem (4) is expressed as:

$$\begin{aligned} \min_x \quad & G(x) \\ s.t \quad & F_i(x) \le 0, \quad i=1,\dots,m \\ & x \in R^n \end{aligned}$$

Where $F_i(x) = \max_{v_i \in V_i} f_i(x,v_i)$, $G(x) = \max_{u \in U} g(x,u)$, F_i, and G denote convex functions. Now, consider the following uncertain optimization problem.

$$\begin{aligned} \min_x \quad & g(x,u) \qquad (5) \\ s.t \quad & H(x,v_1) \le 0 \\ & K(x,v_2) = 0 \\ & x \in R^n \end{aligned}$$

where $H: R^n \to R^{m_1}$ and $K: R^n \to R^{m_2}$ denote vector functions, $g: R^n \to R$ denotes the scalar function of the variable n, $u \in R^p$, $v_1 \in R^{q_1}$, and $v_2 \in R^{q_2}$ represent the uncertain parameters of U, V_1, and V_2 sets, respectively. The optimistic counterpart to the problem (5) is defined as follows:

$$\begin{aligned} \min_x \quad & \min_{(u \in U)} g(x,u) \\ s.t \quad & H(x,v_1) \le 0 \ \textit{for some}\ v_1 \in V_1 \\ & K(x,v_2) = 0 \ \textit{for some}\ v_2 \in V_2 \\ & x \in R^n \end{aligned}$$

If vector x is valid for at least one case of V_1 and V_2 uncertain sets, it is deemed the feasible optimistic solution to the problem (5). The optimistic robust counterpart of the problem (5) is the set of feasible optimistic answers that minimizes the best-case scenario of the target function (i.e., the minimum value of the parameters).

5. Robust counterpart model to determine the most efficient unit

In the following, (3) is associated with its robust counterpart:

$$\begin{aligned} \min\ & d_{\max} \qquad (6) \\ & d_{\max} - d_j \ge 0, & j=1,\dots,n \\ s.t\ & \sum_{i=1}^{m} w_i x_{ij} \le 1 \ \textit{for some}\ x_{ij} \in \eta_{ij}, & j=1,\dots,n \\ & \sum_{r=1}^{s} u_r y_{rj} - u_0 - \sum_{i=1}^{m} w_i x_{ij} + d_j = 0 \\ & \quad \textit{for some}\ x_{ij} \in \eta_{ij}, y_{rj} \in \delta_{rj}, & j=1,\dots,n \\ & \sum_{j=1}^{n} \theta_j = n-1, & j=1,\dots,n \\ & 0 \le d_j \le \theta_j \le 1, & j=1,\dots,n \\ & \theta_j \le N d_j, & j=1,\dots,n \\ & w_i \ge \varepsilon^*, & i=1,\dots,m \\ & u_r \ge \varepsilon^*, & r=1,\dots,s \end{aligned}$$

where η_{ij} and δ_{rj} are the so-called given uncertainty sets.

Theorem 1. If x_{ij} and y_{rj} contain interval uncertainties $\eta_{ij}=\left[\underline{x}_{ij},\overline{x}_{ij}\right]$ and $\delta_{rj}=\left[\underline{y}_{rj},\overline{y}_{rj}\right]$ for $i=1,...,m,\ j=1,...,n,\ r=1,...,s$, then (6) is equivalent to:

$$
\begin{aligned}
&\min d_{\max} && (7)\\
&d_{\max}-d_j\geq 0, && j=1,...,n\\
s.t\quad &\sum_{i=1}^{m} w_i\overline{x}_{ij}\leq 1, && j=1,...,n\\
&\sum_{r=1}^{s} u_r\underline{y}_{rj}-u_0-\sum_{i=1}^{m} w_i\overline{x}_{ij}+d_j\leq 0, && j=1,...,n\\
&-\sum_{r=1}^{s} u_r\overline{y}_{rj}+u_0-\sum_{i=1}^{m} w_i\underline{x}_{ij}-d_j\leq 0, && j=1,...,n\\
&\sum_{j=1}^{n}\theta_j=n-1, && j=1,...,n\\
&0\leq d_j\leq\theta_j\leq 1, && j=1,...,n\\
&\theta_j\leq Nd_j, && j=1,...,n\\
&w_i\geq\varepsilon^*, && i=1,...,m\\
&u_r\geq\varepsilon^*, && r=1,...,s\\
&d_{\max}\geq 0
\end{aligned}
$$

The robust counterpart model 7 in the pessimistic mode, when the inputs are increasing, selects an upper or lower limit on the input and output according to the data, and considering that the above model gives us an efficient unit without ranking, the third and fourth conditions of this If the distance from efficiency becomes zero, the model shows that the increase of output to input is equal to 1, and the unit in question is variable return to scale efficiency. According to the inequality condition, it can be established even for small changes in the data.

Proof. The robust counterpart model (6) is equivalent to the following:

$$
\begin{aligned}
&\min d_{\max} && (8)\\
&d_{\max}-d_j\geq 0, && j=1,...,n\\
&\max_{x_{ij}\in\left[\underline{x}_{ij},\overline{x}_{ij}\right]}\sum_{i=1}^{m} w_i x_{ij}\leq 1, && j=1,...,n\\
&\begin{cases}\min_{y_{rj}=\left[\underline{y}_{rj},\overline{y}_{rj}\right]}\sum_{r=1}^{s} u_r y_{rj}-u_0\\ -\max_{x_{ij}\in\left[\underline{x}_{ij},\overline{x}_{ij}\right]}\sum_{i=1}^{m} w_i\overline{x}_{ij}+d_j\leq 0\end{cases}, && j=1,...,n\\
&\begin{cases}-\max_{y_{rj}=\left[\underline{y}_{rj},\overline{y}_{rj}\right]}\sum_{r=1}^{s} u_r y_{rj}+u_0\\ +\max_{x_{ij}\in\left[\underline{x}_{ij},\overline{x}_{ij}\right]}\sum_{i=1}^{m} w_i\overline{x}_{ij}-d_j\leq 0\end{cases}, && j=1,...,n\\
&\sum_{j=1}^{n}\theta_j=n-1, && j=1,...,n\\
&0\leq d_j\leq\theta_j\leq 1, && j=1,...,n\\
&\theta_j\leq Nd_j, && j=1,...,n\\
&w_i\geq\varepsilon^*, && i=1,...,m\\
&u_r\geq\varepsilon^*, && r=1,...,s\\
&d_{\max}\geq 0
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\max_{x_{ij}\in\left[\underline{x}_{ij},\overline{x}_{ij}\right]}\sum_{i=1}^{m} w_i x_{ij}=\sum_{i=1}^{m} w_i\overline{x}_{ij}\\
&\min_{x_{ij}\in\left[\underline{x}_{ij},\overline{x}_{ij}\right]}\sum_{i=1}^{m} w_i x_{ij}=\sum_{i=1}^{m} w_i\underline{x}_{ij}\\
&\max_{y_{rj}\in\left[\underline{y}_{rj},\overline{y}_{rj}\right]}\sum_{r=1}^{s} u_r y_{rj}=\sum_{r=1}^{s} u_r\overline{y}_{rj}\\
&\min_{y_{rj}\in\left[\underline{y}_{rj},\overline{y}_{rj}\right]}\sum_{r=1}^{s} u_r y_{rj}=\sum_{r=1}^{s} u_r\underline{y}_{rj}
\end{aligned}
\qquad (9)
$$

Now, substituting (9) for (8) yields (7). There is an intriguing aspect to the relationship between the robust counterpart's dual and the uncertain counterpart's dual (3). Thus, the following is the dual of (3):

$$\max \sum_{j=1}^{n} \beta_j - \varepsilon^* \sum_{i=1}^{m} \lambda_i - \varepsilon^* \sum_{r=1}^{s} \sigma_r - \varepsilon^* \sum_{j=1}^{n} r_j + (1-n)\eta$$
$$\sum_{j=1}^{n} \alpha_j = -1, \qquad (10)$$
$$\alpha_j + \gamma_j - Nz_j + t_j \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} \beta_j x_{ij} - \lambda_j - \sum_{j=1}^{n} t_j x_{ij} \le 0, i = 1,\dots,m,$$
$$-\sigma_r + \sum_{j=1}^{n} t_j y_{rj} \le 0, \qquad r = 1,\dots,s,$$
$$-\gamma_j + z_j + r_j + \eta \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} t_j = 0,$$
$$\alpha_j, \beta_j, \gamma_j, z_j, r_j \le 0, \qquad j = 1,\dots,n$$
$$\lambda_i \le 0, \qquad i = 1,\dots,m$$
$$\sigma_r \le 0, \qquad r = 1,\dots,s.$$

Here, (10) is associated with its optimistic robust counterpart as follows:

$$\max \sum_{j=1}^{n} \beta_j - \varepsilon^* \sum_{i=1}^{m} \lambda_i - \varepsilon^* \sum_{r=1}^{s} \sigma_r - \varepsilon^* \sum_{j=1}^{n} r_j + (1-n)\eta$$
$$\sum_{j=1}^{n} \alpha_j = -1, \qquad (11)$$
$$\alpha_j + \gamma_j - Nz_j + t_j \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} \beta_j x_{ij} - \lambda_j - \sum_{j=1}^{n} t_j x_{ij} \le 0, i = 1,\dots,m, \ \text{for some } x_{ij} \in \eta_{ij} = \left[\underline{x}_{ij}, \overline{x}_{ij}\right]$$
$$-\sigma_r + \sum_{j=1}^{n} t_j y_{rj} \le 0, \qquad r = 1,\dots,s, \ \text{for some } x_{ij} \in \delta_{rj} = \left[\underline{y}_{rj}, \overline{y}_{rj}\right]$$
$$-\gamma_j + z_j + r_j + \eta \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} t_j = 0,$$
$$\alpha_j, \beta_j, \gamma_j, z_j, r_j \le 0, \qquad j = 1,\dots,n$$
$$\lambda_i \le 0, \qquad i = 1,\dots,m$$
$$\sigma_r \le 0, \qquad r = 1,\dots,s.$$

Theorem 2. Under interval uncertainties, the duals of the robust counterparts (3) and (11) are as follows:

$$\max \sum_{j=1}^{n} \beta_j - \varepsilon^* \sum_{i=1}^{m} \lambda_i - \varepsilon^* \sum_{r=1}^{s} \sigma_r - \varepsilon^* \sum_{j=1}^{n} r_j + (1-n)\eta$$
$$\sum_{j=1}^{n} \alpha_j = -1, \qquad (12)$$
$$\alpha_j + \gamma_j - Nz_j + t_j \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} \beta_j \overline{x}_{ij} - \sum_{j=1}^{n} \pi_{1j} \overline{x}_{ij} - \lambda_j - \sum_{j=1}^{n} \pi_{2j} \underline{x}_{ij} \le 0, i = 1,\dots,m,$$
$$-\sigma_r + \sum_{j=1}^{n} \pi_{1j} \underline{y}_{rj} - \sum_{j=1}^{n} \pi_{2j} \overline{y}_{rj} \le 0, \qquad r = 1,\dots,s,$$
$$-\gamma_j + z_j + r_j + \eta \le 0, \qquad j = 1,\dots,n,$$
$$\sum_{j=1}^{n} t_j = 0,$$
$$\alpha_j, \beta_j, \gamma_j, z_j, r_j \le 0, \qquad j = 1,\dots,n$$
$$\lambda_i \le 0, \qquad i = 1,\dots,m$$
$$\sigma_r \le 0, \qquad r = 1,\dots,s.$$

Proof. The dual of (7) is identical to (12). Since every real number can be expressed as the difference between two positive real numbers, (11) can be written as $t_j = \pi_{1j} - \pi_{2j}$ so

$$\sum_{j=1}^{n} \beta_j x_{ij} - \lambda_i - \sum_{j=1}^{n} t_j x_{ij} \le 0, \quad i = 1,\dots,m,$$
$$\text{for some } x_{ij} \in \eta_{ij} = \left[\underline{x}_{ij}, \overline{x}_{ij}\right]$$

is written as

$$\sum_{j=1}^{n} \beta_j x_{ij} - \lambda_i - \sum_{j=1}^{n} \left(\pi_{1j} - \pi_{2j}\right) x_{ij} \le 0,$$
$$i = 1,\dots,m, \ \text{for some } x_{ij} \in \eta = \left[\underline{x}_{ij}, \overline{x}_{ij}\right]$$

which is equivalent to

$$-\sigma_r + \sum_{j=1}^{n} t_j y_{rj} \le 0, \ r = 1,\dots,s, \ \text{for some } y_{rj} \in \delta = \left[\underline{y}_{rj}, \overline{y}_{rj}\right] \qquad (13)$$

And also

$$-\sigma_r + \sum_{j=1}^{n} t_j y_{rj} \le 0, \quad r = 1,\dots,s,$$
$$\text{for some } y_{rj} \in \delta = \left[\underline{y}_{rj}, \overline{y}_{rj}\right]$$

is equivalent to

$$-\sigma_r + \min_{y_{rj}\in\delta=\left[\underline{y}_{rj},\bar{y}_{rj}\right]} \sum_{j=1}^{n}\left(\pi_{1j}-\pi_{2j}\right)y_{rj} \le 0 \\ = -\sigma_r + \sum_{j=1}^{n}\pi_{1j}\underline{y}_{rj} - \sum_{j=1}^{n}\pi_{2j}\,\bar{y}_{rj} \le 0, \quad r=1,\dots,s, \tag{14}$$

Now, substituting (13) and (14) for (11) yields (12). As can be seen, the robust model's dual, namely (12), has a significantly smaller number of constraints than the robust model (7). As a result, solving (12) instead of (7) is reasonable.

6. An Illustrative example

A well-known numerical illustration from the DEA literature is used to validate the proposed model. The dataset of 19 facility layout designs (FLDs) In the middle of the imprecise interval with two inputs and four outputs is depicted in Table 1. This section demonstrates the applicability of the proposed model through a numerical example. The example utilizes a dataset of 19 facility layout designs (FLDs) introduced by Ertay et al. [37] and subsequently utilized by Amin and Toloo [36], Amin [4].

The authors employed the proposed model (1) with $\varepsilon = 0.000026$ to determine the most efficient FLD(s) in Table 1. Subsequently, robust model (3) was used to ascertain the most efficient FLD (s) for various interval uncertainties $\eta_{ij} = \left[x_{ij}-\delta, x_{ij}+\delta\right], \sigma_{rj} = \left[y_{rj}-\delta, y_{ij}+\delta\right]$. The results are summarized in Table 2, where the efficient unit (unit 7) remains constant across all interval epsilons. Actual data were used to record the deviation of each unit with a different disorder in this table. As can be seen, the difference is negligible. Despite the deviation and confusion introduced by the inputs and outputs, Table 2 demonstrates that the efficiency deviation variable for unit 7 is zero, indicating that the unit is practically efficient. The deviation of each DMU for various perturbations is depicted in Fig. 1.

Table 1. Inputs and outputs of 19 FLDs in the middle of the imprecise interval.

FLDs	Inputs		Outputs			
	Cost	Adjacency	Shape ratio	Flexibility	Quality	Utility
1	20309.56	6405	0.4697	0.0113	0.0410	30.89
2	20411.22	5393	0.4380	0.0337	0.0484	31.34
3	20280.28	5294	0.4392	0.0308	0.0653	30.26
4	20053.20	4450	0.3776	0.0245	0.0638	28.03
5	19998.75	4370	0.3526	0.0856	0.0484	25.43
6	20193.68	4393	0.3674	0.0717	0.0361	29.11
7	19779.76	2862	0.2854	0.0245	0.0846	25.29
8	19831.00	5473	0.4398	0.0113	0.0125	24.80
9	19608.43	5161	0.2868	0.0674	0.0724	24.45
10	20038.10	6078	0.6624	0.0856	0.0653	26.45
11	20330.68	4516	0.3437	0.0856	0.0638	29.46

FLDs	Inputs		Outputs			
	Cost	**Adjacency**	**Shape ratio**	**Flexibility**	**Quality**	**Utility**
12	20155.09	3702	0.3526	0.0856	0.0846	28.07
13	19641.86	5762	0.2690	0.0337	0.0361	24.58
14	20575.67	4639	0.3441	0.0856	0.0638	32.20
15	20687.50	5646	0.4326	0.0337	0.0452	33.21
16	20779.75	5507	0.3312	0.0856	0.0653	33.60
17	19853.38	3912	0.2847	0.0245	0.0638	31.29
18	19853.38	5974	0.4398	0.0337	0.0179	25.12
19	20355.00	17402	0.4421	0.0856	0.0217	30.02

Table 2. Numerical results under different interval uncertainties

DMU	$\delta = 0$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.001$
1	0.23	0.04	0.05	0.08
2	0.14	0.04	0.05	0.03
3	0.16	0.04	0.04	0.05
4	0.18	0.02	0.02	0.04
5	0.03	0.02	0.02	0.01
6	0.04	0.02	0.03	0.05
7	0.18	0	0	0
8	0.26	0.02	0.02	0.10
9	0.12	0.01	0.01	0.0001
10	0.001	0.03	0.03	0.05
11	0.01	0.03	0.03	0.04
12	0.001	0.02	0.02	0.05
13	0.24	0.01	0.01	0.03

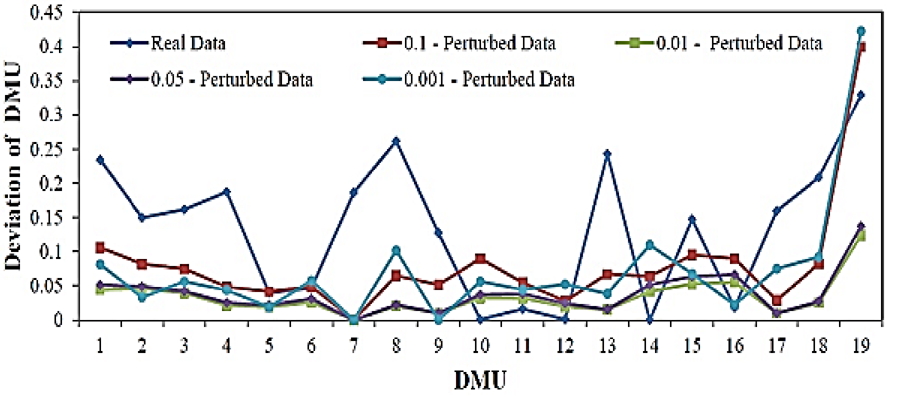

Fig. 1 Deviation of each DMU for different perturbation.

7. concluding remarks

This study presented a novel continuous linear model comprising linear bonds. The advantage of the proposed model over previous models is that it is continuous linear; as a result, the model is effectively solved, and its dual problem is calculable and applicable. Furthermore, the dual problem of robust binaries for the linear problem in the presence of span uncertainty was investigated. The study demonstrated that determining the most applicable deciding unit for a given model in the worst-case scenario was equivalent to gauging the most applicable deciding unit for a dual model in the best-case scenario. This paper aimed to discuss the robust counterpart to the new linear programming (LP) model for identifying the most BCC-efficient decision-making unit for interval uncertainty sets. Moreover, it was demonstrated that the robust problem's dual is identical to the dual problem's optimistic counterpart. The model's capability was demonstrated by its application to In the middle of the imprecise interval data set containing 19 FLPs and by comparing it to the most efficient DMUs in DEA under various uncertainty levels. While various methods exist to provide solution to inexactness in DEA data (e.g. fuzzy DEA models, Imprecise DEA, Interval DEA, stochastic DEA models), the robust DEA (set-based or scenario-based) set its own unique path in characterizing uncertainty and ensuring probability guarantee for reliable efficiency scores, robust discrimination and ranking of DMUs. At the center of the robust DEA is the robust optimization technique which enables us to model uncertainty in the input and output data of DMUs. For the robust DEA to have impact in theory and application, we feel that methodologies that meet the requirements of computational tractability, guarantee for feasibility of the robust DEA solution in terms of uncertainty in both input and output data and feasibility in probability sense if the uncertainty dynamics obey some natural probability distributions are needed. we focused on the set-based model for uncertainty within the context of robust optimization to advance the modeling of the robust DEA. We propose models which satisfy the robust optimization modeling technique and set the basis for robust DEA modeling and applications.

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